TEXT-BOOK

OF

MECHANICS

BY

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KINEMATICS AND KINETICS

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PREFACE

THIS, the second volume of the Text-book of Mechanics, completes an elementary course in Mechanics which, it is hoped, will prepare the student for courses in Applied Mechanics and lay a solid foundation for his future study of more difficult works on Mechanics.

This volume is intended for students possessing a knowledge of the methods of Plane Analytic Geometry and Calculus. It is arranged so that students having a knowledge of the Differential Calculus may undertake its study provided they are pursuing a course in the Integral Calculus.

Besides illustrating the principles of Kinematics and Kinetics the object has been to explain the application of pure mathematics as taught in our schools and thus give the student confidence in its use.

To obtain the best results the student should solve practically all of the many exercises as they occur in the text.

My thanks are again due my wife, Alwynne B. Martin, for many valuable suggestions and aid in reading the proof.

LOUIS A. MARTIN, Jr.

HOBOKEN, N. J., April, 1907.

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KINEMATICS

KINEMATICS

Kinematics treats of the motion of bodies without reference to the forces producing the motion or the masses of the moving bodies. It is thus purely a science of space and time.

KINEMATICS OF A PARTICLE

CHAPTER I

RECTILINEAR MOTION OF A PARTICLE

SECTION I

VELOCITY, ACCELERATION, GRAPHS

Velocity

THE simplest motion a particle can have is motion in a straight line during which equal spaces are passed over in equal times no matter how small the period of time may be taken.

In Fig. 1 a particle is first observed at O_1 when its distance (space) from some reference-point O is s_1 and

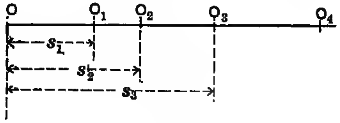


FIG. 1

the time is noted as t_1; the next observation shows the particle at O_2 distant from O by s_2 at time t_2, etc. On comparing the observations it is found that $t_1 < t_2 < t_3$

and $s_1 < s_2 < s_3$, also $\frac{s_2 - s_1}{t_2 - t_1} = \frac{s_3 - s_2}{t_3 - t_2} = \frac{s_4 - s_3}{t_4 - t_3}$, and that the *space increases uniformly.*

This signifies that the change of position ($s_2 - s_1$ or $s_3 - s_2$) divided by the time necessary to make this change ($t_2 - t_1$ or $t_3 - t_2$) is a constant quantity. These fractions designate what is called the *velocity* of the particle.

The change in any quantity divided by the time necessary to produce this change is called the *time-rate of change of this quantity provided the change progresses uniformly.*

Thus *velocity* is *defined as the time-rate of change of position.*

This may be expressed algebraically; thus,

$$\mathbf{v} = \frac{\mathbf{s}_n - \mathbf{s}_m}{\mathbf{t}_n - \mathbf{t}_m} = \frac{\Delta \mathbf{s}}{\Delta \mathbf{t}},$$

provided the velocity is constant.

To obtain the unit of velocity the above equation shows that $\Delta s = s_n - s_m$ and $\Delta t = t_n - t_m$ must both be put equal to 1 so that

$$\text{Unit velocity} = \frac{1 \text{ foot}}{1 \text{ sec.}} = 1 \text{ ft. per sec.}$$

A particle possessing unit velocity would, if the velocity remains constant, pass over 1 foot in each second.

EXERCISE 1. A particle moving in a straight line is observed to be 10 ft., 50 ft., and 159 ft. from a point of reference at 4 hrs. 32 min. 33 sec., 4 hrs. 33 min. 53 sec., and 4 hrs. 37 min. 31 sec., respectively. Find its rate of change of position for each interval. What is its velocity? How far from the point of reference will the particle be at 5 hrs. 37 min. 51 sec.?

EXERCISE 2. Which train has the greater velocity, one moving at the rate of 60 miles per hour or one which travels 100 yards in three seconds?

EXERCISE 3. Find in feet per second the velocity of the earth around the sun, assuming it to describe with constant velocity in 365 days a circle of 92,000,000 miles radius. ($\pi = 22/7$.)

EXERCISE 4. A sprint of 100 yards was accomplished in 12 seconds; what was the sprinter's average velocity in feet per second? In miles per hour?

If the motion of a particle is such that the ratio of the change of position to the corresponding change in time is not constant, then the velocity of the body cannot be constant. The velocity can then no longer be calculated by means of the formula $v = \frac{\Delta s}{\Delta t}$, as this formula can only be applied when the velocity is constant and therefore when the space increases uniformly.

Thus when a train is leaving a station it at first moves slowly, then faster and faster. It would evidently be incorrect to measure the distance the train moves in, say, 10 minutes, and divide by this time to obtain the *starting velocity* or the velocity of the train at the end of the first 10 minutes. Still it is correct to say that the train starts with zero velocity, that at the end of, say, 5 minutes it has a velocity of 8 miles per hour, and that at the end of 10 minutes it has a velocity of 25 miles per hour.

The question as to exactly what is meant by the above statements must be carefully considered. It is well known that when the velocity of the train is given at the end of 5 minutes as 8 miles per hour it is *not* to be understood that the train actually will move 8 miles in the next hour, *but* that *if the velocity of the train were at that*

instant to become constant it would move over 8 miles in the next hour.

To obtain a general expression for the velocity of a particle we turn to the Calculus. There it is shown that $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

As we define velocity as the rate of change of space with respect to time we can write

$$\mathbf{v}=\frac{\mathbf{ds}}{\mathbf{dt}}.$$

This formula holds under all conditions. It does not depend upon the manner in which s varies.

It is well to notice that t, time, is an equicrescent variable, i.e., grows in equal steps. Thus, although time continually increases, it does so uniformly, so that the differential increment, dt, is constant, although time, t, is itself variable.

The method of the Calculus thus affords a perfectly general formula for determining the velocity of a particle moving in any manner whatsoever. Expressed in the language of the Calculus the *velocity of a particle is the first derivative of space with respect to time.*

Example.—A particle moves so that $s=10t-t^2$, where s and t are measured in feet and seconds respectively. What is the velocity of the particle at the start (when $t=0$) and at the end of 6 seconds?

Solution.—As $s=10t-t^2$,

we have $\frac{ds}{dt}=10-2t.$

Therefore $v=10-2t.$

This is the velocity of the particle at any time t.

Thus $\qquad v]_{t=0}=10-2t]_{t=0}=10$ ft. per sec.

and $\qquad v]_{t=6}=10-2t]_{t=6}=-2$ ft. per sec.

The particle thus starts with a velocity of 10 ft. per sec. to the right (indicated by the plus sign), and after 6 seconds it is moving towards the left (indicated by the negative sign) with a velocity of 2 ft. per sec.

EXERCISE 5. By experiment it has been found that a body falling freely from rest in a vacuum near the earth's surface follows approximately the law

$$s=16.1t^2,$$

where s=space (height) in feet, t=time in seconds.

Find the velocity (*a*) at any instant;

(*b*) at end of the first second;

(*c*) at end of the sixth second.

EXERCISE 6. A train left a station and in t hours was at a distance (space) of $s=t^3+2t^2+3t$ miles from the starting-point. Find its velocity (*a*) at the end of t hours; (*b*) at the end of 2 hours.

EXERCISE 7. The space in feet described in t seconds by a point is expressed by the equation $s=48t-16t^2$. Find the velocity at the end of $2\frac{1}{2}$ seconds.

EXERCISE 8. Given $s=2t+3t^2+4t^3$ feet, find the velocity (*a*) at the origin (when $t=0$) and (*b*) at the end of 5 seconds.

EXERCISE 9. Given $s=\frac{a}{t}+bt^2$, where a and b are constants; find the velocity at any instant.

EXERCISE 10. If the distance in feet described by a particle in t seconds is given by the formula $s=5 \log \frac{4}{4+t}$, find the velocity (*a*) at the end of 1 second; (*b*) at the end of 16 seconds.

EXERCISE 11. If the space described is given by $s=ae^t+be^{-t}$, find the velocity at any instant.

EXERCISE 12. Given $s=a\cos\frac{\pi t}{2}$, find the velocity at any instant in terms of s and the constants.

Acceleration

Consider a particle moving in a straight line with varying velocity so that its velocity at the position P_1 is v_1, at which instant the time is t_1; similarly for the positions

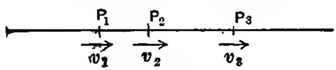


FIG. 2

P_2, P_3, . . . we have v_2, v_3, . . . , and t_2, t_3, . . . (Fig. 2). Assume

$$v_1 < v_2 < v_3, \ldots,$$
$$t_1 < t_2 < t_3, \ldots,$$
$$\frac{v_2 - v_1}{t_2 - t_1} = \frac{v_3 - v_2}{t_3 - t_2} = \frac{v_4 - v_3}{t_4 - t_3} = \ldots,$$

and that the velocity changes uniformly.

Then any increment in the velocity $\Delta v = v_n - v_m$ divided by the corresponding increment in the time $\Delta t = t_n - t_m$ is called the *acceleration of the particle*, which in this case is a constant.

$$\therefore\ \mathbf{a} = \frac{\mathbf{v}_n - \mathbf{v}_m}{\mathbf{t}_n - \mathbf{t}_m} = \frac{\Delta \mathbf{v}}{\Delta \mathbf{t}}.$$

Acceleration is the *time-rate of change of velocity*. Unit acceleration is obtained by putting $v_n - v_m = 1$ foot per second, and $t_n - t_m = 1$ second, or $a = \frac{1 \text{ foot per second}}{1 \text{ second}}$ $= 1$ foot-per-second per second.

EXERCISE 13. The velocity of a body at 2 min. 3 sec., 2 min. 33 sec., 3 min. 53 sec. is observed to be 15 feet per sec., 30 feet per sec., 70 feet per sec., respectively. Find its acceleration and the velocity it would have at (*a*) 5 min. 10 sec. (*b*) 5 min. 30 sec.

EXERCISE 14. A particle moving from rest with a constant acceleration has a velocity of 160 feet per second after 5 seconds; find its acceleration and its velocity after 7 seconds.

EXERCISE 15. A particle moving with a negative acceleration of 32 feet-per-sec. per sec. is projected with a velocity of 160 feet per sec. Find when it will come to rest and what will be its velocity after 10 seconds.

EXERCISE 16. A train starts from rest and after 1 minute its velocity is 30 miles per hour. Find the acceleration in feet-per-second per second, supposing it constant.

Assume now that the velocity no longer increases uniformly; the acceleration therefore becomes variable and the formula $a=\frac{\Delta v}{\Delta t}$ then no longer gives the acceleration.

Here again we make use of the Calculus and put

$$\mathbf{a}=\frac{\mathbf{dv}}{\mathbf{dt}},$$

for the acceleration is always the time-rate of change of the velocity.

Thus: *Acceleration is always the first derivative of velocity with respect to time.*

Example.—A particle moves so that $s=c\cos(kt)$, where c and k are constants. Deduce an expression for its acceleration at any time t.

What will be the acceleration when $t=4$ seconds if $c=2$ and $k=3$?

Solution.—As $$s=c\cos(kt),$$

$$\frac{ds}{dt}=-ck\sin(kt);$$

$$\therefore\ v=-ck\sin(kt);$$

also $$\frac{dv}{dt}=-ck^2\cos(kt);$$

$$\therefore\ a=-ck^2\cos(kt),$$

which is the first answer sought.

Now if $c=2$, $k=3$, and $t=4$, then

$$a=-(2)(3)^2\cos(3)(4)=-18\cos(12).$$

As 12 radians $=12\left(\frac{180}{\pi}\right)=688$ degrees,

$$\cos 12=\cos 688°=\cos 32°=0.85,$$

and $$a=-18(0.85)=-15.3 \text{ ft.-per-sec. per sec.}$$

The minus sign shows that the velocity is decreasing.

Exercise 17. Find the accelerations of the particles the motions of which are described in Exercises 5 to 12.

Exercise 18. If a point moves in a fixed path so that $s=\sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity.

Exercise 19. In t hours a train has reached a point at a distance of $\frac{1}{4}t^4-4t^3+16t^2$ miles from the starting-point. (*a*) Find its velocity and acceleration. (*b*) When will the train stop to change the direction of its motion? (*c*) Describe the motion during the first 10 hours.

It has been demonstrated that under all conditions $v=\frac{ds}{dt}$ and $a=\frac{dv}{dt}$. Substituting the value of v in that of a and noting that t is equicrescent so that dt is a constant, we have

$$a=\frac{d\left(\frac{ds}{dt}\right)}{dt}=\frac{\frac{d^2s}{dt}}{dt}=\frac{d^2s}{dt^2}.$$

$\frac{d^2s}{dt^2}$ is called the second derivative of s with respect to t.

The formulæ to be remembered are

$$\mathbf{v=\frac{ds}{dt}}, \quad \mathbf{a=\frac{dv}{dt}}, \quad \text{and} \quad \mathbf{a=\frac{d^2s}{dt^2}}.$$

Another very useful form into which the differential expression for acceleration can be thrown is $\mathbf{a=v\frac{dv}{ds}}$.

This form is obtained as follows:

$$a=\frac{dv}{dt}=\frac{dv}{dt}\frac{ds}{ds}=\frac{ds}{dt}\frac{dv}{ds}=v\frac{dv}{ds}.$$

Exercise 20. By the means of the formula $a=v\frac{dv}{ds}$, find the acceleration if $s=5b^t$, where b is a constant.

To the present only the methods of finding the velocity and acceleration, when the law of variation of space is known, have been considered.

The reverse operation of finding the velocity and space traversed when the law of variation of the acceleration is given must now occupy our attention.

As this problem is the reverse of that already studied

it follows that instead of applying the principles of the Differential Calculus, and thus passing from given functions to their derivatives, a reverse process must be found for passing from the derivatives back to the functions themselves. This branch of mathematics is known as the Integral Calculus.

To comprehend this more fully, consider the function

$$y=x^3+x^2+3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and its differential

$$dy=3x^2dx+2x\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now assume that the original function (1) is unknown and it is proposed to pass from (2) back to some function not involving differentials.

Considering each term of (2) separately it is evident that dy could only have been obtained by differentiating y; similarly $3x^2dx$ at once suggests x^3 and $2x\,dx$ suggests x^2. So that we would write $y=x^3+x^2$ as the integral expression of (2). On considering (1) this result is seen to be only partly true, for the *constant term is lacking.* The reason for this discrepancy becomes apparent when we remember that the differential of a constant is zero. It is thus seen that to every *result of integration there should be appended a constant term.* The integral of (2) becomes $y=x^3+x^2+C$, where C represents any constant.

The value of this constant cannot be determined from a purely mathematical standpoint, but when the integration is applied to a physical problem the conditions of the problem supply the necessary data for its determination.

This is illustrated by the following

Example.—Find the position of a particle at a given time, t, when the velocity varies as the distance from a given point on the rectilinear path, and if at the time $t=0$ the particle is at a distance s_0 from the given point.

Solution.—In Fig. 3 consider O the given point from which the space is to be measured and P the position of

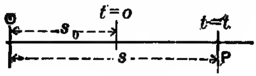


FIG. 3

the particle at any time, t; then $v \propto s$ or $v=ks$, where k is a constant. As $v=\frac{ds}{dt}$, we have $\frac{ds}{dt}=ks$.

Before attempting the integration of any expression all variables of one kind should be placed in one member of the equation, thus: $\frac{ds}{s}=k\,dt$. Now searching amongst the rules for differentiation for an expression of the form of $\frac{ds}{s}$ and concluding that it can only be obtained by the differentiation of a logarithm, put $\log s=kt+C$, where C is the constant of integration.

To determine the constant C return to the question and note that when $t=0$, $s=s_0$; substituting these values in the integral expression we have $\log s_0=k(0)+C$. $\therefore$ $C=\log s_0$ and the complete integral is

$$\log s=kt+\log s_0.$$

To simplify this transpose all logarithmic expressions to one member of the equation, thus:

$$\log s - \log s_0 = kt, \quad \text{or} \quad \log\left(\frac{s}{s_0}\right) = kt.$$

$\therefore \ \dfrac{s}{s_0} = \varepsilon^{kt}$, by the definition of a logarithm, and $s = s_0 \varepsilon^{kt}$.

EXERCISE 21. Find the position of a particle whose velocity varies as the time if the space is measured from the position of the particle when the time is zero.

Example.—How far will a particle move during the fourth second of its motion if the velocity at the end of the third second is 81 ft. per sec. and if the velocity varies as the square of the time?

Solution.—As the velocity varies as the square of the time, we write

$$v \propto t^2,$$

or $$v = kt^2,$$

where the constant k can be determined from the statement that $$v = 81$$

when $$t = 3.$$

$$\therefore \ 81 = k(3)^2,$$

or $$k = 9.$$

So that $$v = 9t^2,$$

but $$\frac{ds}{dt} = v,$$

$$\therefore \ ds = 9t^2 dt$$

and $$s = 3t^3 + C.$$

If we measure s from the position of the particle when $t = 0$, then $s = 0$ when $t = 0$; so that

$$0=3(0)^3+C,$$

$$\therefore\ C=0$$

and $$s=3t^3.$$

Now let s_3 and s_4 represent the spaces separating the particle from its position at $t=0$ when t equals 3 and 4 seconds respectively. Then

$$s_4=3(4)^3=192$$

and $$s_3=3(3)^3=\ 81,$$

$$\therefore\ s_4-s_3=111 \text{ feet},$$

which is the space traversed by the particle during the fourth second of its motion.

EXERCISE 22. How far will a particle move in 2 minutes when the velocity is 5 ft. per sec. at the end of the first second and varies as the time?

EXERCISE 23. Find the space described by a particle in 5 seconds, assuming that it commences to move with a velocity of 30 ft. per sec. and has an acceleration of 10 ft.-per-sec. per sec.

EXERCISE 24. The velocity of a particle at the end of the first 10 feet of its motion is 30 ft. per sec. How far will it be at the end of 5 seconds from a point of reference 20 feet from its position at the beginning of these 5 seconds if the velocity varies as the space measured from this reference-point?

EXERCISE 25. Find the velocity and space at the end of the time, t, when the acceleration varies as the time from the position of rest.

Graphs

SPACE-TIME CURVES

Whenever two magnitudes bear a certain relation to each other, when one is a function of the other, this relationship can be represented by means of a plane curve. This plane curve is called the graph of the function.

Let Fig. 4 represent the graph of the relation between space and time for the motion of a particle. Here we

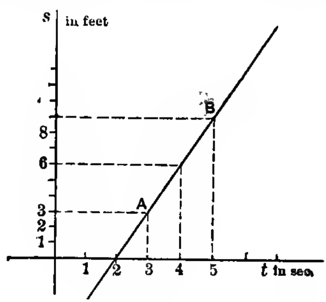


FIG. 4

lay off the values of time, t, as abscissæ and the corresponding values of space, s, as ordinates. Thus when $t=4$ seconds we see that $s=6$ feet. This graph shows that the space increases as the time increases, and as the graph is a straight line the space increases uniformly, so that the velocity of the particle is constant.

To obtain the velocity take any two points on the graph (say A and B) and measure the corresponding increments of space and time ($\Delta s=9-3=6$ feet and $\Delta t=5-3=2$ seconds); these give the velocity

$$v = \frac{\Delta s}{\Delta t} = \frac{6}{2} = 3 \text{ ft. per sec.}$$

If the space-time curve is not a straight line, the velocity can still be obtained from the graph. This is illustrated in Fig. 5. Here the space-time curve is curved so that the velocity continually changes. Suppose the velocity of the particle at a certain time t_1 is sought, then at the corresponding point P_1 on the curve a tangent is drawn. If the velocity of the particle at the time t_1 suddenly becomes uniform, the graph would become the tangent P_1T and the velocity would be $v = \frac{\Delta s}{\Delta t}$; but this is precisely what is meant by the velocity of the particle at the time t_1 (see page 7). So that if we find the ratio of an increment of space to a corresponding increment of time *measured on a tangent* to the space-time curve, we obtain the velocity of the particle at the time corresponding to the point of tangency.

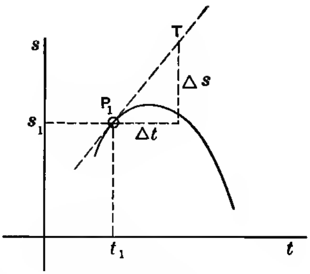


FIG. 5

If the same scales are used to measure space and time then $\frac{\Delta s}{\Delta t}$ *corresponds to the slope of the tangent,* or it is the trigonometric tangent of the angle which the geometric tangent to the space-time curve makes with the axis of time.

EXERCISE 26. What is the difference between the velocities deduced from the space-time curve shown in Fig. 6 (a) and (b)?

EXERCISE 27. What can be said of the velocity of a particle whose space-time curve is shown in Fig. 6 (c)?

EXERCISE 28. Find the velocity of the particle in Ex. 27 when t equals 2, 4, 5, and 6 seconds respectively.

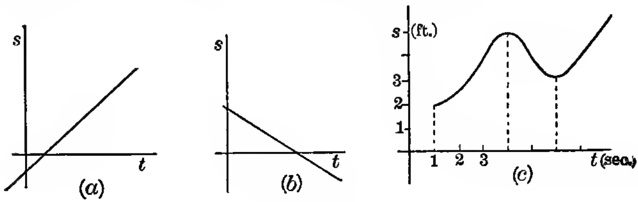


FIG. 6

EXERCISE 29. What would a space-time curve consisting of a straight line parallel to the s-axis represent? Parallel to the t-axis?

VELOCITY-TIME CURVES

In a manner similar to the above we may plot a graph representing the relation between the velocity and the time of a moving particle. If the graph is a straight line (Fig. 7) the velocity changes uniformly and the acceleration is constant. In Fig. 7

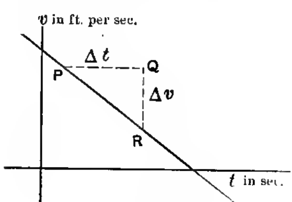


FIG. 7

$$a = \frac{\Delta v}{\Delta t} = \frac{-RQ}{PQ} = -\frac{RQ}{PQ}.$$

If the graph is curved, then the acceleration varies from point to point and is determined by the slope of the tangent to the curve at the point under consideration, as already explained for the velocity.

EXERCISE 30. What difference is there between the acceleration deduced from the velocity-time curves shown in Fig. 8 (*a*) and (*b*)?

EXERCISE 31. What is the acceleration of the particle whose velocity-time curve is shown in Fig. 8 (*c*) at the times corre-

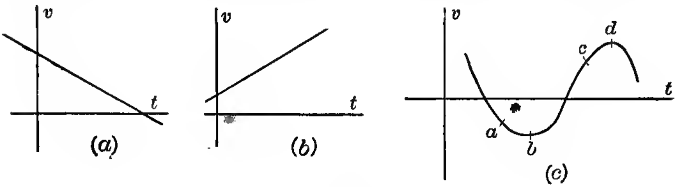


FIG. 8

sponding to *a*, *b*, *c*, and *d* if the time scale is 1 inch = 8 seconds and the velocity scale is 1 inch = 40 feet per sec.?

EXERCISE 32. Draw a velocity curve for a particle whose acceleration is positive and constant between $t=0$ and $t=1$, then gradually diminishes to zero at $t=2$, then gradually increases negatively.

EXERCISE 33. Plot the space-time, velocity-time, and acceleration-time curves for the motion indicated by the equation $s=16t^2$.

EXERCISE 34. Plot the space-time curve for $s=\frac{1}{4}t^3$ between $t=-3.5$ and $t=+3.5$, and from this obtain *graphically* the velocity-time and acceleration-time curves.

SECTION II

APPLICATIONS TO SPECIAL CASES

Rectilinear Motion with Constant Acceleration

In Fig. 9 assume A to be the position of a particle at the time $t=0$ and B its position at any subsequent time $t=t$. Let the acceleration of the particle be a, its initial

velocity v_0, and let its velocity at the time t be v, while the distance between its initial position and its position

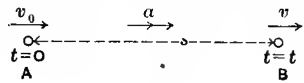


FIG. 9

at the time t is represented by s. Here a and v_0 are constant, while v, s, and t are variable.

It is proposed to find the relations existing between the quantities a, v, v_0, s, and t by means of the differential equations

$$a=\frac{d^2s}{dt^2}=\frac{dv}{dt}=v\frac{dv}{ds}.$$

Put $a=\dfrac{dv}{dt}$; then $dv=adt$ and $v=at+C_1$, where C_1 is the constant of integration to be determined by the given condition that v is v_0 when t is 0.

Thus, $v_0=a(0)+C_1$. $\therefore\ C_1=v_0$,

$$\text{or}\quad v=at+v_0.$$

As $v=\dfrac{ds}{dt}$ we may now put

$$\frac{ds}{dt}=at+v_0;\quad\text{then}\quad ds=at\,dt+v_0dt,$$

and integrating we obtain $s=\dfrac{at^2}{2}+v_0t+C_2$, where C_2, another constant of integration, is to be determined by the condition that s is 0 when t is 0. Thus,

$$0=\frac{a(0)^2}{2}+v_0(0)+C_2,\quad\therefore\ C_2=0.$$

So that $$s=\frac{at^2}{2}+v_0t.$$

Again, starting with $a=v\frac{dv}{ds}$ we have $v\,dv=a\,ds$; integrating we have $\frac{v^2}{2}=as+C_3$, wherein C_3 may be found, as we know v to be v_0 when $s=0$.

Thus, $$\frac{v_0^2}{2}=a(0)+C_3 \quad \text{and} \quad C_3=\frac{v_0^2}{2}.$$

Whence $$\frac{v^2}{2}=as+\frac{v_0^2}{2}, \quad \text{or} \quad v^2=2as+v_0^2.$$

To recapitulate, the principal formulæ (to be carefully committed to memory) are

$$\left.\begin{aligned} \mathbf{v}&=\mathbf{at}+\mathbf{v_0}, \\ \mathbf{s}&=\frac{\mathbf{at^2}}{2}+\mathbf{v_0t}, \\ \text{and} \quad \mathbf{v^2}&=\mathbf{2as}+\mathbf{v_0^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By the proper interpretation of these formulæ all problems relating to the rectilinear motion of any particle moving with constant acceleration may be solved.

Unless otherwise specified the quantities s, v, v_0, and a will be assumed as positive if measured or directed towards the right.

Exercise 35. A particle is moving with an acceleration of -5 ft.-per-sec. per sec. If it started at $t=0$ with a velocity of 20 ft.-per-sec., find its velocity and position 2, 3, 4, 5, 7, 8, and 10 seconds after starting, and draw separate diagrams to

scale showing its position and direction of motion for each of the above specified times. The diagrams should be placed so that the initial point of each lies on the same vertical line.

Note that s is *not necessarily* the space passed over by the particle. What does it represent?

EXERCISE 36. Same as Ex. 35 if the initial velocity is -10 ft. per sec. and the acceleration 5 ft.-per-sec. per sec.

EXERCISE 37. Same as Ex. 35 if the initial velocity is -10 ft. per sec. and the acceleration -5 ft.-per-sec. per sec.

EXERCISE 38. A body moves with an acceleration of 3 ft.-per-sec. per sec. and starts with a velocity of 13 ft. per sec. What is its velocity after having moved 50 feet? How long does it take to perform the journey?

EXERCISE 39. A body is known to move with a constant acceleration of -10 ft.-per-sec. per sec. What does this statement mean? If the velocity of the body at some instant is 100 ft. per sec., what is its velocity after 40 feet have been described, and how much farther must it go before the velocity is reduced to 10 ft. per sec.? When will it come to rest? If the conditions remain unchanged, will it remain at rest? If not, what will be its subsequent motion?

EXERCISE 40. A stone skimming on ice passes a certain point with a velocity of 20 feet per second and suffers a retardation of 1 ft.-per-sec. per sec. Find the space described in the next 10 seconds, and the whole space described when the stone has come to rest.

EXERCISE 41. A steamer starting from rest was observed to move 200 ft. in 30 seconds and 500 ft. in the next 30 seconds. How many seconds elapsed before the time of the first period of observation, assuming the acceleration constant?

It has been experimentally demonstrated that particles whose motion near the surface of the earth is unimpeded move with a constant acceleration of g (approximately

32) ft.-per-sec. per sec. Therefore the equations of motion (1) may be applied to such bodies.

EXERCISE 42. Assuming the upward direction as positive, rewrite equations (1), adapting them to a particle:

(*a*) dropped from rest;

(*b*) projected vertically downward with a velocity V;

(*c*) " " upward " " " V.

EXERCISE 43. If a body is projected vertically upward with a velocity V, find:

(*a*) the time during which the body rises;

(*b*) the time of flight before returning to the starting-point;

(*c*) the greatest height to which the particle will rise;

(*d*) the velocity of the body on returning to the starting-point.

EXERCISE 44. A bullet is fired vertically from a rifle and leaves the muzzle of the rifle with a velocity of 1000 ft. per sec. How high will it go (neglecting effect of air resistance), and in how many minutes will it return to earth?

EXERCISE 45. A balloon is rising uniformly with a velocity of 10 ft. per sec., when a stone dropped from it reaches the ground in 3 seconds; find the height of the balloon (1) when the stone was dropped, (2) when it reached the ground.

EXERCISE 46. If a body after having fallen for 3 seconds breaks a pane of glass, and thereby loses one third of its velocity, find the entire distance through which it will fall in 4 seconds.

EXERCISE 47. To what height will a body projected upward with a velocity of 48 ft. per sec. rise? At what times after starting will it be 32 ft. from the ground?

EXERCISE 48. A stone is projected vertically upward from the top of a vertical cliff with a velocity of 128 ft. per sec. If the cliff is 80 feet high, how long will it take the stone to reach the foot of the cliff?

EXERCISE 49. Find the velocity with which a body must be projected downward to overtake in 4 seconds a body which has fallen from rest from the same point through 16 feet.

Simple Harmonic Motion

Simple Harmonic Motion can be defined as the motion of the projection of a point moving in a circular path and describing equal arcs in equal intervals of time; the projection to be made upon any straight line in the plane of the circle.

Thus in Fig. 10 assume the point P to describe the circle in 8 seconds in such a manner that it moves over

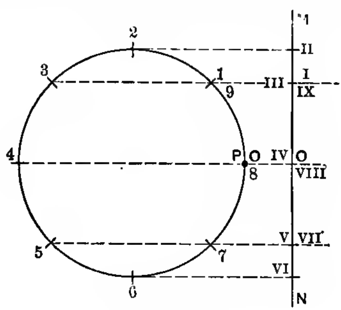


FIG. 10

one eighth of the circumference per second; then the projection of P upon the line MN will describe the path II VI with a simple harmonic motion (S.H.M.). If the point starts at P when $t=0$, it will after successive intervals of one second occupy the positions 0, 1, 2, 3, . . . , while its projection will at corresponding times be at O, I, II, III, . . .

A point in S.H.M. possesses a rectilinear motion of

an oscillating nature, returning to its initial point after equal intervals of time. S.H.M. is a periodic function of time.

The circle used in the definition of S.H.M. is called the "circle of reference"; it can be advantageously used in the study of S.H.M.

In connection with Fig. 11, where p_0 represents the initial position of the particle (when $t=0$) and p any subsequent position (when $t=t$), note the following definitions:

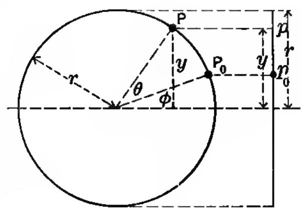


FIG. 11

The *Phase* of the particle is the angle $(\phi+\theta)$.* It is the angle through which the corresponding particle in the circle of reference has moved since the particle in S.H.M. passed through the mid-point of its path in the positive direction.

The *Lead* (or *Lag* if negative) is the angle ϕ.* It is the phase of the particle when $t=0$.

The *Displacement* is y. It is the distance of the particle from the mid-point of its path.

The *Amplitude* is r, the radius of the circle of reference.

The *Period* is the time of one revolution of P or of one complete oscillation of p.

The *Frequency* is the number of revolutions of P per second, or the number of oscillations of p per second.

EXERCISE 50. A particle in S.H.M. has a period of 18 seconds, an amplitude of 10 feet, and lags by $\frac{\pi}{6}$. Compute its

* All angles are to be measured in radians (see p. 53).

phases and displacements 1, 1.5, 5, 10.5, and 16 seconds after starting. Illustrate your results in a sketch.

EXERCISE 51. Compute the frequency of the particle described in Ex. 50.

The velocity and the acceleration of a particle in S.H.M. at any phase of its motion will now be considered.

In Fig. 12 let v represent the velocity of the particle in S.H.M. This will be the projection of V, the velocity of the imaginary particle, in the circle of reference.

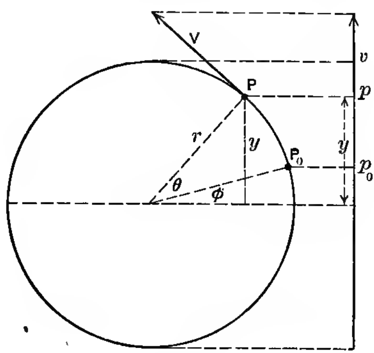


FIG. 12

Let T be the period, n the frequency; then

$$n=\frac{1}{T},$$

and as all angles are measured in radians, and $2\pi n$ (which we will denote by ω) is the number of radians described by P per second,

$$\theta=2\pi nt=\frac{2\pi t}{T}=\omega t.$$

From Fig. 12 we see that

$$y = r\sin(\theta+\phi)$$
$$= r\sin(\omega t+\phi),$$

where r, ϕ, and ω are constants.

To find the velocity we differentiate, and noting that

$$v = \frac{dy}{dt}$$

we have $$v = r\omega\cos(\omega t+\phi).$$

Also, as $$a = \frac{dv}{dt},$$

we have $$a = -r\omega^2\sin(\omega t+\phi),$$

or $$\mathbf{a} = -\omega^2\mathbf{y}.$$

As ω is a constant and y is the displacement we can state that:

The acceleration of a particle in S.H.M. is proportional to its displacement and is always directed towards the mid-point of its path.

This is an important characteristic of S.H.M.

Solve the following exercises *without using formulæ.*

EXERCISE 52. A particle in S.H.M. has an amplitude of 2 feet and a period of $\frac{\pi}{2}$ seconds. Calculate its velocity and acceleration. What is its velocity and acceleration for its maximum positive and negative elongations (distances from its central position)?

EXERCISE 53. What velocity does a particle in S.H.M. possess at the ends of its path? What acceleration at the mid-point of its motion?

EXERCISE 54. At what points of its path are the velocity and the acceleration at the maximum or minimum?

EXERCISE 55. Show that the period of a particle in S.H.M. is equal to

$$2\pi\sqrt{-\frac{\text{displacement at any point}}{\text{acceleration at that point}}}.$$

EXERCISE 56. If the frequency and amplitude of a particle in S.H.M. are 1/12 and 10 feet respectively, what is the phase and displacement 8 seconds after a passage through an extreme positive elongation?

EXERCISE 57. What are the velocity and acceleration of the particle in Ex. 56 at the instant considered and also when the displacement is 8 feet?

SECTION III

RELATIVE MOTION

Composition and Resolution of Velocities and Accelerations

Velocities have both direction and magnitude, they are *vector quantities*. They can therefore be graphically represented by straight lines whose directions and lengths represent the directions and magnitudes of the velocities.

Forces are also vector quantities, and all theorems concerning the composition and resolution of forces are equally applicable to velocities.

As an example consider the following illustration of the parallelogram of velocities:

In Fig. 13 assume MN to represent a flat car, and A a man walking upon it with a velocity represented by AB. If the man continued walking with a constant velocity AB, he would in one second find himself at B.

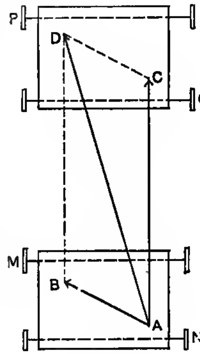


FIG. 13

Assume now that the man does not walk upon the car, but that the car moves with a velocity represented by AC. If the car now continues uniformly for one second, it will occupy the position PQ, and the man would be at C.

If both motions occur simultaneously, then the man will in one second find himself at D, a vertex of the parallelogram $ABCD$, and the man's actual velocity (the resultant velocity due to the components AB and AC) will be AD, the diagonal of the parallelogram.

EXERCISE 58. Velocities of 6 ft. per sec. and 10 ft. per sec. are impressed upon a particle. Find the resultant velocity if the angle between the velocities is 90°. What are the greatest and least values of the resultant, the angle between them being varied at will?

EXERCISE 59. A particle has a velocity of 15 ft. per sec. which is resolved into two components at right angles. The magnitude of one component is 9 ft. per sec.; find that of the other.

EXERCISE 60. The resultant of two velocities of 3 ft. per sec. and 5 ft. per sec. respectively is a velocity of 7 ft. per sec. Find the angle between the two velocities.

EXERCISE 61. A particle has velocities of 3, 3, and 5 units,

inclined at 120° to each other, impressed upon it. Find the resultant velocity and its direction.

EXERCISE 62. Find the vertical velocity of a train when moving up a 1% grade at 40 miles per hour.

Accelerations are also vector quantities. Therefore the composition and resolution of accelerations can be performed, as already explained for forces and velocities.

Relative Motion

The position of a particle is determined by its distance from some point of reference. As the point of reference is assumed at will, the position of the particle is relative to the point of reference. Thus, the path of a moving particle is always referred to a set of axes. These axes are considered fixed, that is, are assumed to remain at rest. Observations show that nothing in the universe is at rest, absolutely at rest. This does not prevent the assumption of rest for the axes of reference, but it should be remembered that the path of the particle is not its absolute path, but only its *path relative to the assumed axes*, or relative to the body upon which the axes are drawn.

Consider, for instance, the path of a point on the rim of the wheel of a moving wagon. Relative to a set of axes on the wheel the point does not move, its path is a point; relative to a set of axes on the body of a wagon the path of the point is a circle; and relative to a set of axes on the earth the path is a cycloid. Ordinarily the earth is considered at rest and all motion relative to it is denoted as "absolute."

In finding the velocity of a body, A, relative to another body, B, it is only necessary to subtract the velocity of B from that of A, for this is equivalent to considering B at rest.

To illustrate this consider a bullet rising with a velocity u and a balloon ascending with a velocity v. It is required to find the velocity with which the bullet strikes the balloon, i.e., the velocity of the bullet relative to the balloon. Assume the balloon and bullet both ascending through a medium moving downward with a velocity v, then the absolute velocity of the balloon will be zero and the relative velocity of the bullet will not be changed, for the motion of the medium affects both bodies equally. The bullet will under the assumed condition possess an absolute velocity of $u—v$, and as the balloon is now at rest, this is the relative velocity required.

Thus reversing the velocity of the body relative to which the velocity is to be considered and adding it to the velocity of the other body gives the required relative velocity of that body.

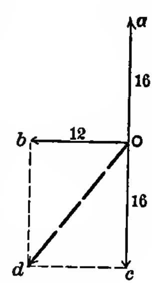


FIG. 14

Example.—A steamer is sailing north at 16 miles per hour in an east wind blowing 12 miles per hour. Find the apparent direction and velocity of the wind to a passenger on the steamer.

Solution.—Fig. 14 illustrates the problem. As the velocity of the wind relative to the steamer is required, reverse the velocity of the steamer oa, this gives oc; add this to ob, the velocity of the wind. Hence the apparent velocity of the wind is $od=20$ miles per hour, and its apparent direction is $\tan^{-1} \frac{3}{4}$ east of north.

EXERCISE 63. Two vessels start at the same time from the same harbor, one sailing east at 12 miles per hour and the other south at 9 miles per hour. Find the velocity of the first relative to that of the second. What is the velocity of the second relative to that of the first?

EXERCISE 64. A boat is propelled at 12 miles per hour across a stream flowing at 5 miles per hour in a direction perpendicular to the current. Find the "absolute velocity" of the boat.

EXERCISE 65. A vessel steams due north at 10 miles per hour. The apparent direction of the wind is N. 30° W., and its apparent velocity is 30 miles per hour. Find the actual velocity of the wind.

EXERCISE 66. A man walking 3 miles per hour meets a carriage moving along the same road at 5 miles per hour. Find the apparent velocity of the man to a person in the carriage if they are moving in (*a*) the same direction, (*b*) in opposite directions. (*c*) What would be the velocity of the man relative to the carriage if they are moving over straight roads intersecting at right angles?

EXERCISE 67. The hood of a market-van is 3.5 feet above the floor. In driving through a shower the floor is wet to a distance of 11 inches behind the front edge of the hood. Assuming the rain-drops to fall vertically with a constant velocity of 28 feet per second, find the velocity of the van in miles per hour.

EXERCISE 68. A steamer is moving east with a velocity of 6 miles per hour; the wind appears to blow from the north. The steamer increases its velocity to 12 miles per hour, and the wind now appears to blow from the northeast. What is the true direction and velocity of the wind?

EXERCISE 69. A train moving at 30 miles per hour is struck by a stone moving horizontally and at right angles to the track with a velocity of 33 feet per second. With what velocity does the stone strike the train?

CHAPTER II

CURVILINEAR MOTION OF A PARTICLE

In the treatment of curvilinear motion the velocities and accelerations are resolved into components. Either axial components or tangential and normal components are usually employed.

Section IV

AXIAL COMPONENTS OF VELOCITY AND ACCELERATION

These components are parallel to the two rectangular axes to which the path of the particle is referred.

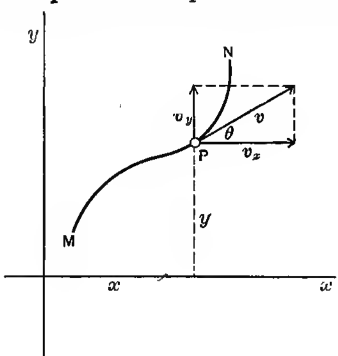


Fig. 15

In Fig. 15 let MN be the path of the particle, and v its velocity at the point P; then v_x and v_y will be the axial components of its velocity.

As v is tangent to the path at P, and ds is an element of the path,

$$\sin\theta=\frac{dy}{ds} \quad\text{and}\quad \cos\theta=\frac{dx}{ds};$$

and as $v=\frac{ds}{dt}$, $v_x=v\cos\theta$, and $v_y=v\sin\theta$, it follows that

$$\mathbf{v}_x=\frac{\mathbf{dx}}{\mathbf{dt}} \quad\text{and}\quad \mathbf{v}_y=\frac{\mathbf{dy}}{\mathbf{dt}}.$$

If a_x and a_y be used to denote the axial components of the acceleration, then as a_x and a_y are the rates of change of v_x and v_y respectively, and we have

$$\mathbf{a}_x=\frac{\mathbf{dv}_x}{\mathbf{dt}}=\frac{\mathbf{d^2x}}{\mathbf{dt^2}}$$

and

$$\mathbf{a}_y=\frac{\mathbf{dv}_y}{\mathbf{dt}}=\frac{\mathbf{d^2y}}{\mathbf{dt^2}}.$$

The magnitude and direction of the actual velocity of the particle is

$$v=\frac{ds}{dt}=\sqrt{v_x^2+v_y^2} \quad\text{and}\quad \tan^{-1}\frac{v_y}{v_x}.$$

Similarly for the actual acceleration we have

$$a=\sqrt{a_x^2+a_y^2} \quad\text{and}\quad \tan^{-1}\frac{a_y}{a_x}.$$

Example.—The parametric equations of the path of a particle are

$$\left.\begin{array}{l} x=a\cos t,\\ y=b\sin t,\end{array}\right\}\text{where } a \text{ and } b \text{ are constants.}$$

Find the rectangular equation of the path, the magnitude and direction of the velocity and acceleration at any time.

Solution.—The parametric equations of the path involve in addition to the variables x and y a third variable, t. To find the rectangular equation of the path this third variable must be eliminated.

Thus, $\cos t=\frac{x}{a}$, $\therefore\ \sin t=\sqrt{1-\cos^2 t}=\frac{\sqrt{a^2-x^2}}{a}$, and

$$y=b\sin t=\frac{b}{a}\sqrt{a^2-x^2},$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the rectangular equation.

To find the velocity we have, from

$$x=a\cos t \quad \text{and} \quad y=b\sin t,$$

$$v_x=\frac{dx}{dt}=-a\sin t \quad \text{and} \quad v_y=\frac{dy}{dt}=b\cos t.$$

$$\therefore\ v=\sqrt{a^2\sin^2 t+b^2\cos^2 t}=\sqrt{\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}}=\frac{\sqrt{a^4y^2+b^4x^2}}{ab},$$

and if θ is the angle its direction makes with the x-axis,

$$\tan\theta=\frac{v_y}{v_x}=-\frac{b\cos t}{a\sin t}=-\frac{b^2x}{a^2y}.$$

To find the acceleration we have

$$a_x=\frac{dv_x}{dt}=-a\cos t \quad \text{and} \quad a_y=\frac{dv_y}{dt}=-b\sin t,$$

so that $$a=\sqrt{a^2\cos^2 t+b^2\sin^2 t}=\sqrt{x^2+y^2}.$$

And if ϕ be its inclination to the x-axis,

$$\tan\phi=\frac{a_y}{a_x}=\frac{b\sin t}{a\cos t}=\frac{y}{x}.$$

EXERCISE 70. From the results in the preceding example find the velocity and acceleration if $a=3$ feet, $b=6$ feet, and $t=\frac{\pi}{2}$ seconds. Draw a diagram showing the results.

From the above it should be noted that the acceleration and the velocity of the particle do not have the same direction; this must evidently be the case, as otherwise the motion would be rectilinear.

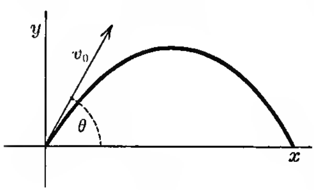


FIG. 16

In curvilinear motion the velocity can even be constant and still there must be an acceleration. *The acceleration then changes the direction of the velocity* without changing its magnitude.

EXERCISE 71. Find the rectangular equation of the curve whose parametric equations are:

$$a\begin{cases}x=a(t-\sin t),\\ y=a(1-\cos t);\end{cases}\qquad b\begin{cases}x=vt\cos\theta,\\ y=-\dfrac{gt^2}{2}+vt\sin\theta.\end{cases}$$

EXERCISE 72. Neglecting the resistance of the air, the equations of the path of a projectile (Fig. 16) are $x=v_0t\cos\theta$ and $y=v_0t\sin\theta-16t^2$, where v_0=initial velocity, θ=angle of

projection t=time of flight in seconds; x and y being measured in feet. Find the magnitudes and directions of the velocity and acceleration (a) at any instant; (b) at the end of the first second, having given $v_0=100$ ft. per sec., $\theta=30°$.

EXERCISE 73. If a point referred to rectangular coordinates moves so that $\begin{cases} x=r\cos t+b \\ y=r\sin t+c \end{cases}$, show that its velocity is constant. What is its acceleration?

EXERCISE 74. If the path of a moving point is the sine curve $\begin{cases} x=ct \\ y=b\sin ct \end{cases}$, show ($a$) that the x component of the velocity is constant; (b) that the acceleration of the point at any instant is proportional to its distance from the axis of X. Is the motion parallel to the y-axis simple harmonic?

SECTION V

TANGENTIAL AND NORMAL COMPONENTS OF VELOCITY AND ACCELERATION

In this system the components are taken along the tangent and normal to the path of the particle at the position of the particle at the instant considered.

As the velocity of a particle is always along a tangent to the path, the tangential component of the velocity is the velocity itself, and the normal component is zero.

Thus if v_t and v_n represent the tangential and normal components of the velocity v, we have

$$\mathbf{v}_t=\mathbf{v}=\frac{\mathbf{ds}}{\mathbf{dt}} \quad \text{and} \quad \mathbf{v}_n=0,$$

where ds is an element of the path.

Consider now the components of the acceleration. In Fig. 17 let PT and PN represent the tangent and the normal at P, a the actual acceleration of the particle at P, and a_x, a_y its axial components.

To find a_t and a_n the tangential and normal accelerations respectively, resolve a_x and a_y into components

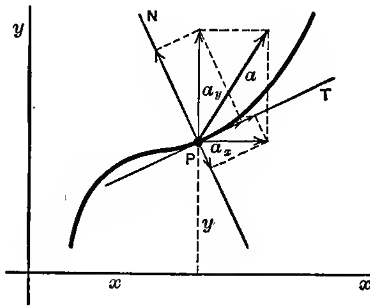


FIG. 17

along PT and PN. If α is the inclination of PT to the x-axis, then

$$a_t = a_x \cos\alpha + a_y \sin\alpha$$

and $$a_n = a_y \cos\alpha - a_x \sin\alpha.$$

As $\cos\alpha = \dfrac{dx}{ds}$ and $\sin\alpha = \dfrac{dy}{ds}$, where ds is the element of the path of the particle, we have

$$a_t = \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds}$$

and $$a_n = \frac{d^2y}{dt^2}\frac{dx}{ds} - \frac{d^2x}{dt^2}\frac{dy}{ds}.$$

These expressions may be greatly simplified in the following manner:

Consider first the expression for a_t; here we may put

$$a_t = \frac{d^2x}{dt^2}\frac{dx}{dt}\frac{dt}{ds} + \frac{d^2y}{dt^2}\frac{dy}{dt}\frac{dt}{ds} = \left\{ \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} \right\} \frac{dt}{ds},$$

and as $v^2 = v_x^2 + v_y^2$, we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2;$$

by differentiation we obtain

$$2\left(\frac{ds}{dt}\right)\frac{d^2s}{dt^2} = 2\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + 2\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right).$$

$$\therefore\ a_t = \left\{ \left(\frac{ds}{dt}\right)\frac{d^2s}{dt^2} \right\} \frac{dt}{ds} = \frac{d^2s}{dt^2} = \frac{dv}{dt}.$$

Note carefully that in curvilinear motion, although v is always equal to $\frac{ds}{dt}$, a, the acceleration, is not equal to $\frac{d^2s}{dt^2}$ or $\frac{dv}{dt}$. These expressions give only the tangential component of the actual acceleration.

For the simplification of a_n it is necessary to introduce the radius of curvature (ρ) of the path at P.

The value of ρ is usually remembered as

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

where x is the independent variable. For the present investigation t should be the independent variable. To change the independent variable note that

$$\left(\frac{dy}{dx}\right)=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}},$$

and that

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3},$$

whence by substitution

$$\rho=\frac{\left\{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}=\frac{\left(\dfrac{ds}{dt}\right)^3}{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}=\frac{v^3}{v_xa_y-v_ya_x}.$$

Returning to the expression for a_n we may write

$$\begin{aligned}a_n&=\frac{d^2y}{dt^2}\frac{dx}{dt}\frac{dt}{ds}-\frac{d^2x}{dt^2}\frac{dy}{dt}\frac{dt}{ds}\\&=\left\{\frac{d^2y}{dt^2}\frac{dx}{dt}-\frac{d^2x}{dt^2}\frac{dy}{dt}\right\}\frac{dt}{ds}\\&=\left\{\left(\frac{ds}{dt}\right)^3\frac{1}{\rho}\right\}\left(\frac{dt}{ds}\right)\\&=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2=\frac{1}{\rho}v^2.\end{aligned}$$

Thus, $$\mathbf{a}_t=\frac{\mathbf{d^2s}}{\mathbf{dt^2}}=\frac{\mathbf{dv}}{\mathbf{dt}} \quad \text{and} \quad \mathbf{a}_n=\frac{\mathbf{v}^2}{\rho}.$$

Notice that $a_t=\dfrac{d^2s}{dt^2}$ is not the whole acceleration but only its tangential component.

Example.—Find the acceleration of a particle moving in a circular path of radius r with a constant velocity $\boldsymbol{v}$.

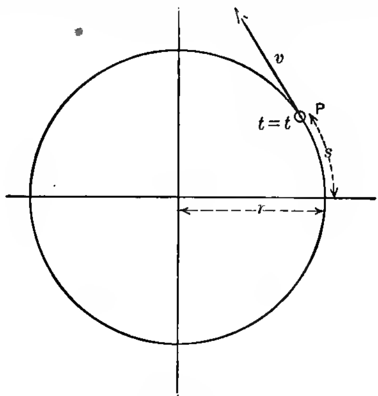


FIG. 18

Solution.—Consider the particle at the point P at any time t (Fig. 18).

Here $$v_t=v=\frac{ds}{dt} \quad \text{and} \quad v_n=0.$$

Also $$a_t=\frac{dv}{dt}=0, \quad \text{as } v \text{ is constant.}$$

To find a_n note that the radius of curvature of a circle at any point is constant and equal to r; thus,

$$a_n=\frac{v^2}{\rho}=\frac{v^2}{r}.$$

As the tangential acceleration is zero the resultant acceleration must be along the normal and equal to a_n. Also, a_n is positive; therefore a_n must be directed toward the center of curvature.

The acceleration of the particle is thus equal to $\frac{v^2}{r}$ and acts towards the center of the circle.

Query.—Assuming the moon's orbit to be circular, what is meant by the "moon falling towards the earth"?

EXERCISE 75. A particle moves in a circle of 10 feet radius, so that the arc described equals four times the cube of the time. Find v_t and v_n and thus find the velocity 2 seconds after leaving the end of a horizontal diameter.

EXERCISE 76. Calculate a_t and a_n and thus find the acceleration of the particle of Ex. 75.

Why is this acceleration not directed towards the center of the circle?

EXERCISE 77. Compute v_x and v_y and thus v for Ex. 75. Compare result with that of Ex. 75.

Hint.—Find x in terms of θ, the angle described by the radius drawn to the particle; then θ in terms of s and then in terms of t. Similarly for y.

EXERCISE 78. Compute a_x and a_y and thus find a for Ex. 75. Compare result with that of Ex. 76.

EXERCISE 79. A body moves so that $v_x = 12t$ and $v_y = 4t^2 - 9$; find x, y, the equation of the path; v, v_n, v_t; a_x, a_y, a; a_n, a_t, a; the radius of curvature of the path, and the length of the path, s. Plot the results for $t = 3$ seconds, assuming the origin of coordinates at the position of the particle when $t = 0$.

SECTION VI

COMPOSITION OF SIMPLE HARMONIC MOTIONS

The methods for finding the resultant motion of a particle subjected to two S.H.Ms. will now be considered.

Example.—To find the resultant of two S.H.Ms. of equal amplitudes at right angles to one another, with a phase difference of $\frac{\pi}{2}$ and of the same period.

Graphical Solution.—In Fig. 19 assume *gc* to be the path of the particle *P* when subject to one of the S.H.Ms.;

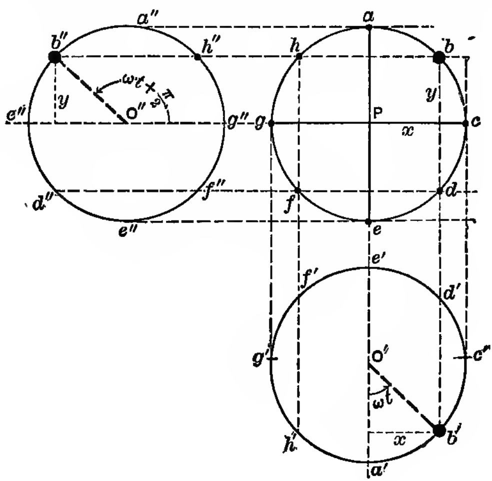


FIG. 19

then the circle O' is its circle of reference. Let *ea* be the path of *P* when under the influence of the second S.H.M. represented by the circle O''. Assume the particle at *P* and moving towards *c* owing to its S.H.M.,

represented by the circle O'; the corresponding point then is a', and its phase is zero. When the position of the particle with regard to the motion represented by the circle O'' is at g'' its phase is zero. But the phases of the S.H.Ms. should differ by $\frac{\pi}{2}$; therefore the position on the circle O'' simultaneous with a' is at a'', 90° in advance of g''.

Divide the circles O' and O'' into any convenient number of parts and letter the simultaneous positions a', a''; b', b'', etc. Then the actual position of the particle under both motions will be found at the intersections of the projecting lines at a, b, etc.

The resultant motion is thus seen to be in a circle of radius equal to the amplitudes of the S.H.Ms. and in the direction *abc*.

Analytical Solution.—Consider the lines *gc* and *ae*, Fig. 19, as the x and y axes of coordinates. The x and y displacements of P are then given by

$$x = r \sin \omega t$$

and

$$y = r \sin\left(\omega t + \frac{\pi}{2}\right),$$

where the letters have the same significance as in Section II, page 29.

These equations are the parametric equations of the resultant path of the particle. From them, by the elimination of ωt, we obtain

$$x^2 + y^2 = r^2,$$

the rectangular equation of the resultant path.

EXERCISE 80. Find the magnitude and direction of the resultant velocity and acceleration of the particle in the above example assuming the period as $\frac{1}{8}$ second, the amplitude as 4 feet, and $t=\frac{1}{96}$ second.

EXERCISE 81. Find the resultant of two S.H.Ms. having equal amplitudes at right angles to one another, equal periods, and

(*a*) being in the same phase;

(*b*) the vertical S.H.M. lagging by $\frac{\pi}{2}$;

(*c*) the vertical S.H.M. lagging by π.

EXERCISE 82. Find the resultant of two S.H.Ms. having equal amplitudes at right angles, equal periods, but differing in phase by $\frac{\pi}{6}$. (Divide circles of reference into 24 equal parts.)

EXERCISE 83. Same as Exercise 82 but one amplitude being twice the other.

In case the *periods of the S.H.Ms. to be compounded differ* it is simply necessary to remember that the angle described by the corresponding point in the circle of reference varies inversely as the period.

Example.—Find the resultant motion compounded of two S.H.Ms. having equal amplitudes at right angles but whose periods are as four to five and which differ in phase by $\frac{\pi}{2}$.

Solution.—This is illustrated in Fig. 20. The vertical S.H.M., circle O'', has a period $\frac{5}{4}$ of the horizontal S.H.M., circle O'; therefore O'' is divided into 20 parts, while O' is divided into only 16 parts, as the angle at

the center of O'' must be $\frac{4}{5}$ of the corresponding angle at the center of O'.

Assuming a' as the starting-point of S.H.M. O' (phase

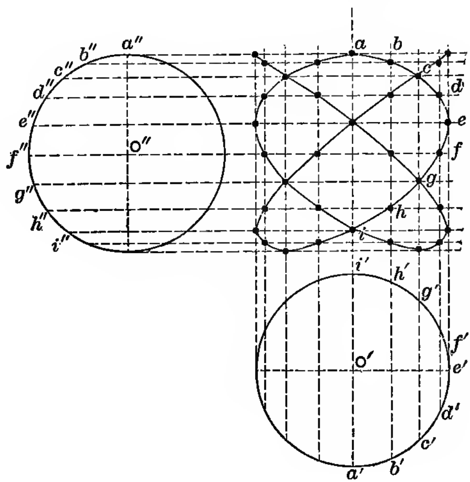


FIG. 20

being zero), a'' will be the corresponding point of S.H.M. O'' (phase being $\frac{\pi}{2}$ in advance of a'). The actual motion of the particle may then readily be followed as a, b, c, etc.

EXERCISE 84. Find the resultant of two S.H.Ms., the amplitude of the vertical one being $\frac{1}{2}$ of that of the horizontal one, and the period of the vertical S.H.M. being $\frac{2}{3}$ of that of the horizontal S.H.M., their phases being the same.

EXERCISE 85. Same as Exercise 84 if the vertical S.H.M. leads by $\frac{\pi}{4}$.

EXERCISE 86. Same as Exercise 84 if the vertical S.H.M. leads by $\frac{\pi}{2}$.

EXERCISE 87. Same as Exercise 84 if the vertical S.H.M. leads by $\frac{3\pi}{4}$.

So far only the composition of S.H.Ms. at right angles have been considered. For the composition of S.H.Ms. in the same line the *Harmonic Curve* is used.

Assume a particle executing a S.H.M. along the path *MN* (Fig. 21) with *C* as medial position and *O* as circle

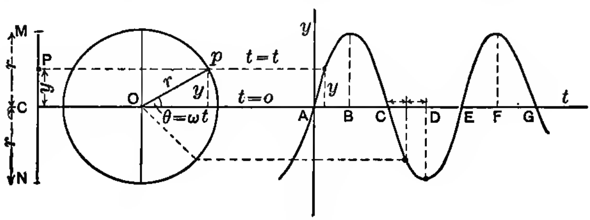


FIG. 21

of reference. Let the particle be at *C* and moving upward at $t=0$, while *P* is its position at $t=t$ when y is its displacement. Then $y=r\sin\theta=r\sin\omega t$. But $\omega=\frac{2\pi}{T}$, where T is the period; therefore $y=r\sin\left(\frac{2\pi}{T}\right)t$.

Plot the curve $y=r\sin\left(\frac{2\pi}{T}\right)t$, using the horizontal line as t-axis and a vertical line as y-axis. If $t=0$, then $y=0$. If $t=\frac{T}{4}=AB$, then $y=r\sin\left(\frac{\pi}{2}\right)=r$; if $t=\frac{T}{2}=AC$, then $y=0$; if $t=\frac{3T}{4}=AD$, then $y=-r$. Intermediate

points may be found in a similar way and the *harmonic curve* thus plotted.

Thus it is seen that the harmonic (also called the sine) curve shows at a glance the displacement of the particle in S.H.M. at any time. How would you plot this curve graphically?

EXERCISE 88. Plot the harmonic curve showing the displacements of a particle having an amplitude of 5 feet and a period of 2 seconds.

The composition of S.H.Ms. in the same line is effected by plotting the harmonic curves corresponding to each S.H.M. and then adding the displacements at the same time, as shown by these curves, to obtain the displacement of the resultant motion.

Example.—Find the resultant of two S.H.Ms. in the same line if the amplitude and period of one are both twice those of the other and their phases are the same.

Solution.—In Fig. 22, AB shows the path of the par-

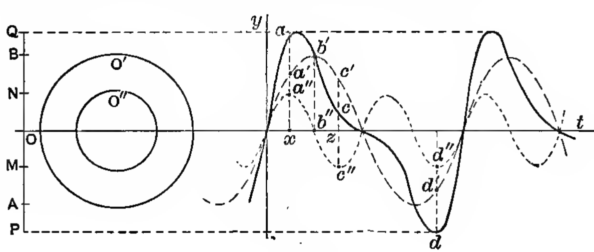


FIG. 22

ticle executing one S.H.M., MN the path of the particle executing the other S.H.M.; the circles of reference are O' and O'', and the corresponding harmonic curves $a'b'c'$, etc., and $a''b''c''$, etc.; the resultant curve is abc, etc.

The distance $ax = a''x + a'x$ and $cz = c'z - c''z$, etc.

A careful study of the resultant harmonic curve will show that the particle under the influence of the combined S.H.Ms. moves over the path PQ in the following manner: Starting at O it moves upward to Q, then downward to O, moving very slowly as it passes through O, then downward to P, whence rapidly upward through O. The particle now repeats this cycle, so that its motion is still periodic although not harmonic.

EXERCISE 89. Find the resultant of three S.H.Ms. along the same line, their amplitude being to each other as $1:\frac{1}{2}:\frac{1}{4}$, while their periods are as $1:\frac{1}{2}:\frac{1}{3}$ respectively, the first and third being in the same phase, while the second differs from them by π.

EXERCISE 90. Find the resultant of three S.H.Ms. in the same line, their amplitude being to each other as $1:\frac{1}{3}:\frac{1}{5}$, while the frequences are in the ratio $1:3:5$ and all having the same phase.

Exercises 89 and 90 will prepare the reader to accept the statement that by the proper choice of a number of harmonic curves, their amplitudes, periods, and phases, and by compounding these any periodic curve of any complexity may be built up, provided the curve nowhere goes to an infinite distance from the t-axis and the periods of the component motions are commensurate. This fact was first developed by Fourier, and the statement is usually known as *Fourier's Theorem.*

KINEMATICS OF A RIGID BODY

CHAPTER III

MOTION OF A RIGID BODY

THE bodies now to be considered are rigid, i.e., the relative position of the particles of which the bodies are composed are not supposed to change.

The motion of bodies will be considered under three heads: 1. Translation; 2. Rotation; 3. Plane Motion in General.

SECTION VII

TRANSLATION AND ROTATION

Translation

Translation may be defined as a motion in which the particles of the body all describe equal parallel paths in the same time.

From this definition it is evident that if a body possesses a motion of translation only, the motion of any one of its particles fully describes the motion of the body. Therefore the principles already discussed for the solution of problems relating to the motion of particles suffice for the solution of problems relating to a motion of translation of bodies.

Rotation

Rotation may be defined as the motion of a body in which the particles move in circles the centers of which lie in a fixed straight line called the *axis of rotation.*

If the axis of rotation passes through the body, the particles on it may be considered as moving in circles of zero radius and they are therefore at rest.

The planes of all the circles described by the particles of the body are perpendicular to the axis of rotation. Any one of these planes may be called the plane of rotation.

In Fig. 23, A_1, B_1, C_1 represent particles of a body rotating about O, and A_2, B_2, C_2 the same particles at some subsequent time.

From Fig. 23 it is evident that the paths described by the various particles in a given time are not of the same length. But if the body is rigid the *angles described in a given time by the various particles of the body are equal.* Thus in considering the rotation of a body it is usual to deal not with the distance described by any portion but with the angle through which the whole body has turned. This angle is equal to the angle through which any particle has turned. It is called the angular displacement of the body. The angular displacement is considered positive if counter-clockwise.

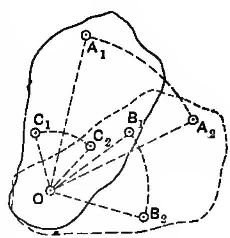


FIG. 23

In mechanics, as in higher mathematics, *angles are* exclusively *measured in radians.* This measure of angles can be deduced from the following definition.

An angle is measured by the ratio of its intercepted arc to the radius of this arc. Thus in Fig. 24 if θ represents the angle ABC, s the length of its intercepted arc, and r the radius of the arc, then

$$\theta = \frac{s}{r}.$$

As the angle changes the length of the arc, s, also changes, but the radius, r, remains constant. Thus to obtain the unit angle, i.e., $\theta = 1$, the arc s must be made equal to r, for only then can $\theta = \frac{s}{r}$ be equal to unity.

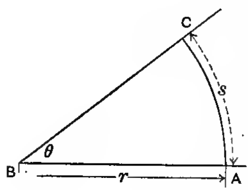


FIG. 24

The angle subtended by an arc equal to the radius is the unit angle and this unit has been named the *radian*.

When an angle is designated as 3 radians it means that the angle subtends an arc whose length is three times its radius.

Radians may readily be converted into degrees by noting that an angle of 360° would intercept an arc whose length is $2\pi r$ and the angle of 360° would thus be measured by $\frac{2\pi r}{r} = 2\pi$ radians.

Or we may state that

An angle of $360° = 2\pi$ radians;

$\therefore$ " " " $1° = \frac{2\pi}{360}$ radians $= 0.0174533$ radian,

and 1 radian = an angle of $\left(\frac{360}{2\pi}\right)^{\circ} = 57.29578° = 57°17'45''$.

In mathematics it is usual to omit the word radian in stating the value of an angle.

EXERCISE 91. Find by means of trigonometric tables sin $\frac{1}{2}$; tan 1; sin 3; cos 5; tan 100.

Angular velocity is defined as the time-rate of change of the angular displacement. It is usually designated by ω.

Thus $\omega = \frac{\theta_n - \theta_m}{t_n - t_m} = \frac{\Delta\theta}{\Delta t}$ if the angular velocity is constant, or $\omega = \frac{d\theta}{dt}$ if the angular velocity is variable.

The *unit of angular velocity* is obtained if the rotating body describes an angle of 1 radian in 1 second, for then is

$$\omega = \frac{1 \text{ radian}}{1 \text{ second}} = 1 \text{ radian per second.}$$

The sign of ω is considered positive if the body turns in the counter-clockwise direction.

EXERCISE 92. The radius to a particle of a rotating body at 2.55 P.M. is observed to make an angle of 45° with a horizontal line, at 2.57 P.M. its angle is 360135°. Find the angular velocity of the body.

EXERCISE 93. If a rotating body makes 1350 revolutions per minute, what is its angular velocity?

EXERCISE 94. A body rotates with a constant angular velocity of 3 radians per second; how many revolutions per minute does it make?

EXERCISE 95. The angle through which a body turns is given by the equation $\theta = 10t^2 + 5$. Is its angular v locity constant? What is its angular velocity when $t=0$? when $t=10$ sec.?

Angular Acceleration is defined as the time-rate of change of angular velocity. It is usually designated by α.

Thus $\alpha = \frac{\omega_n - \omega_m}{t_n - t_m} = \frac{\Delta\omega}{\Delta t}$ or $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$.

The unit of angular acceleration is obtained if the angular velocity increases by 1 radian per second in 1 second, for then

$$\alpha = \frac{1 \text{ radian per sec.}}{1 \text{ second}} = 1 \text{ radian-per-sec. per sec.}$$

The angular acceleration is positive or negative as the angular velocity increases or decreases.

EXERCISE 96. If the angular velocity is observed to be 10 radians per sec. and 3 seconds later is 4 radians per sec., what is the angular acceleration of the body?

EXERCISE 97. A body rotates in such a manner that $\theta = \sin \pi t$. Find its angular velocity and acceleration. Find the time when the body is momentarily at rest and find the angular acceleration at these times.

EXERCISE 98. Plot and interpret the "space-time", "velocity-time", and "acceleration-time" curves for the body whose motion is described in Exercise 97.

If a body rotates with a constant angular acceleration, α, and at the time $t=0$ possesses an angular velocity ω_0, then the relations existing between these quantities and θ and ω, the angle through which it has turned, and its angular velocity at any time t are given by

$$\left.\begin{aligned} \omega &= \alpha t + \omega_0 \\ \theta &= \frac{\alpha t^2}{2} + \omega_0 t \\ \omega^2 &= 2\alpha\theta + \omega_0^2 \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

These equations should be compared with equations (1), page 23.

EXERCISE 99. Starting with the differential values for angular acceleration, deduce equations (2).

EXERCISE 100. A bodymakes n revolutions per second. Find its angular velocity.

EXERCISE 101. A wheel starts from rest at 1 P.M. Find its angular velocity at 1.10 P.M. if it rotates with a constant angular acceleration of $\frac{1}{2}$ rad.-per-sec. per sec. How many revolutions did it make in this time?

EXERCISE 102. A wheel making 50 revolutions per second is brought to rest with a uniform angular acceleration in 10 seconds. How many turns does it make before coming to rest?

EXERCISE 103. A wheel making 20 revolutions per second is gradually brought to rest after making 100 revolutions. What was its angular retardation?

SECTION VIII

THE VELOCITY AND ACCELERATION OF ANY POINT IN A ROTATING BODY

In Fig. 25, let O be the axis of rotation and P any point in the body. Let the radius OP be r, the arc P_0P be s, and the angular displacement be θ. Then

$$s=r\theta,$$

$$\therefore \frac{ds}{dt}=r\frac{d\theta}{dt};$$

or $$v=r\omega.$$

Also,
$$\frac{d^2s}{dt^2}=r\frac{d^2\theta}{dt^2},$$

or
$$a_t=r\alpha.$$

But
$$a_n=\frac{v^2}{r}=\frac{\omega^2r^2}{r}=\omega^2r,$$

$$\therefore\ a=r\sqrt{\alpha^2+\omega^4}.$$

The results to be remembered are

$$\mathbf{v}=\omega\mathbf{r},\quad \mathbf{a}_t=\alpha\mathbf{r},\quad \text{and}\quad a_n=\omega^2\mathbf{r}=\frac{\mathbf{v}^2}{\mathbf{r}}.$$

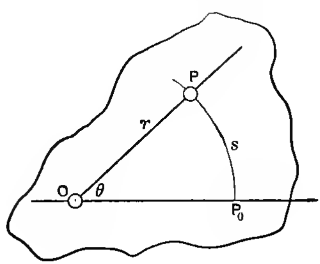


FIG. 25

EXERCISE 104. A belt passes over a pulley d feet in diameter and making n revolutions per minute. Find the linear velocity of the belt.

EXERCISE 105. A wheel 6 feet in diameter revolves 420 times per minute. Find its angular velocity and the linear velocity of a point 1.5 feet from the center. What is the acceleration of this point?

EXERCISE 106. A coin, radius r, rolls along a table. Find its angular velocity if the linear velocity of its center is v; find the linear velocity of the highest and lowest points in its circumference.

SECTION IX

PLANE MOTION IN GENERAL

Plane Motion is defined as a motion in which each point of the body remains at a constant distance from a fixed plane. This plane or any plane parallel to it is called the plane of motion.

EXERCISE 107. Name some cases in which a moving body possesses plane motion. Can rotation be included under the head of plane motion?

The plane motion of a body is determined if the motions of any two of its points are known. If one of these points be fixed the motion becomes one of rotation about this point. If both points be fixed the body is at rest.

Plane Motion Considered as a Combined Rotation and Translation

In plane motion all displacements may be considered as consisting of a rotation and a translation. Thus in

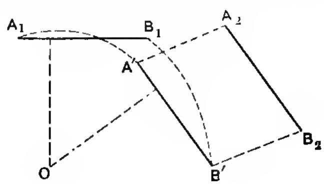


FIG. 26

Fig. 26, if A_1B_1 be the initial and A_2B_2 the final positions of two points of a body, we may regard this dis-

placement to have occurred by rotation about any point O, due to which A_1B_1 moves to $A'B'$, and then by translation from $A'B'$ to A_2B_2.

EXERCISE 108. (*a*) Select any point above A_1B_1 (Fig. 26) as the center of rotation and show how the displacement from A_1B_1 to A_2B_2 may then be separated into a rotation and a translation. (*b*) Similarly, using A_1 as center. (*c*) Similarly, using B_1 as center.

Any plane motion can thus be considered as a rotation about any point and a simultaneous translation. In Fig. 27 let O and P be any two points in a body. Let O be taken as the projection of the axis of rotation on the plane of motion. Then the point P will in general possess a motion composed of a rotation about O and of a translation possessed by every point of the body.

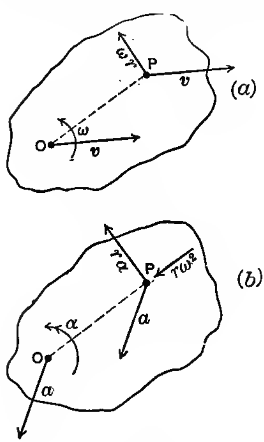


FIG. 27

Let ω and α be the angular velocity and angular acceleration, respectively, of the body with respect to O; let v and a be the velocity and acceleration of the translation. Then, as shown in Fig. 27 (*a*), O will have a velocity equal to v only; while P possesses a velocity whose components are v (due to the translation) and ωr (due to the rotation about O). In Fig. 27 (*b*) the accelerations of O and P are shown. O possesses only the acceleration of translation a. P possesses besides this accelera-

tion two other accelerations due to the rotation about O, the tangential acceleration αr and the normal acceleration $\omega^2 r$ directed as shown.

EXERCISE 109. A body possesses a motion of translation defined by $v = 10$ ft. per sec. directed towards the east and $a = 5$ ft.-per-sec. per sec. north. It also possesses about a point O in the body a rotation defined by $\omega = -3$ rad. per sec. and $\alpha = 2$ rad.-per-sec. per sec. Calculate the actual velocities and accelerations of the points $\left(6, \frac{2\pi}{3}\right)$ and $\left(4, \frac{\pi}{4}\right)$ referred to the point O and the radius directed towards the east as the initial line.

EXERCISE 110. A rod 4 feet long possesses a velocity of translation of 10 ft. per sec. at right angles to its length and an angular velocity of 10 rad. per sec. about its center. Find the actual velocities of the ends and center of the rod. What point in the rod would be momentarily at rest? Could the motion of the rod be considered as a pure rotation?

EXERCISE 111. A wheel, radius r, rolls upon horizontal ground with a constant angular velocity ω. Find the velocity of any point on the rim. What will be the velocity of the highest point on the rim?

Notice that the arbitrary point selected as center of rotation possesses only a velocity of translation, for a rotation about a point cannot affect its velocity. Therefore if any point be selected as axis of rotation its actual velocity must be the velocity of translation of the body.

Consider a wheel rolling along horizontal ground. (Draw a diagram.) Let the velocity of the center be constant and equal to v. If we consider the point of contact with the ground as the center of rotation, then the velocity of translation will be zero, for we know that this

center of rotation has no velocity. If ω be the angular velocity of the wheel about the selected center of rotation, then $\omega = \frac{v}{r}$.

Now, although the velocity of the center of the wheel, v, and ω are constant, this does *not* imply that there exists no acceleration. Of course, as ω is constant there can be no angular acceleration, and as the center of the wheel moves in a *straight line* with constant velocity the center of the wheel can have no acceleration. But, owing to ω, the center of the wheel will possess an acceleration $\omega^2 r$ directed vertically downward and the only way to neutralize this acceleration is to impart to the whole wheel an acceleration of translation equal and oppo ite to the acceleration imposed by ω upon the center of the wheel. Thus the motion of the wheel can be regarded as a rotation about the point of contact with the ground with a constant angular velocity, $\frac{v}{r}$, and zero angular acceleration combined with a translation having no velocity and an acceleration of $\omega^2 r$ directed vertically upward.

EXERCISE 112. The center of a wheel (radius 2 feet) rolling on horizontal ground possesses an acceleration of 3 ft.-per-sec. per sec. If it starts from rest what will be the velocities and accelerations of points on the rim 2 feet and 4 feet above the ground after 10 seconds?

Solve this exercise by using (*a*) the point of contact with the ground, (*b*) the right-hand end of a horizontal diameter, as a center of rotation.

Instantaneous Rotation

Motion of Connected Points.—In Fig. 28 (*a*) consider the velocities v_1 and v_2 of the points P_1 and P_2, which are rigidly connected so that the line P_1P_2 cannot change its length. The velocities v_1 and v_2 can each be resolved into two components, one parallel and the other perpendicular to the line P_1P_2, as shown. As, by hypothesis, the distance P_1P_2 cannot change, the components of the velocities along P_1P_2 (i.e., P_1M and P_2N) must be equal in magnitude and have the same direction.

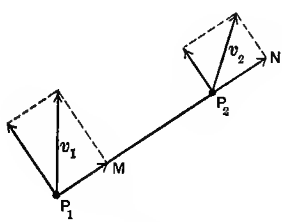


FIG. 28 (*a*)

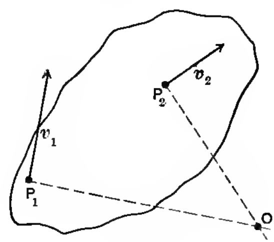


FIG. 28 (*b*)

Plane motion may also be considered as an *instantaneous rotation* about an *instantaneous axis* without additional translation. That this is so becomes evident on considering Fig. 28 (*b*). Here P_1 and P_2 represent any two points of a body whose velocities are respectively v_1 and v_2. If we draw P_1O perpendicular to the direction of the velocity v_1, we can assert that no point on this perpendicular will possess a component velocity along P_1O, for P_1 possesses no such component and all points along the line P_1O are rigidly connected to P_1. Similarly for all points along P_2O. Therefore O can have no

component velocities along either OP_1 or OP_2. It must therefore remain at rest. Thus O is a center of rotation for the motions of P_1 and P_2 and therefore for the whole body at the instant under consideration.

The line in a moving body (or in its extension), all points of which at a certain instant have no velocity, is called the *instantaneous axis* of the body and its foot in the plane of motion is the *instantaneous center*.

In general this instantaneous center continually changes its position. The locus of its successive positions is known as a *centrode*.

Note carefully that the velocities v_1 and v_2, Fig. 28 (*b*), are not independent of one another. If ω be the angular velocity of the body about the instantaneous center, O and $OP_1=r_1$, and $OP_2=r_2$; then

$$v_1=r_1\omega \quad \text{and} \quad v_2=r_2\omega,$$

so that
$$\omega=\frac{v_1}{r_1}=\frac{v_2}{r_2}.$$

Also the velocity, v, of any other point P whose distance from O is r, would be

$$v=r\omega=r\frac{v_1}{r_1},$$

and its direction would be perpendicular to OP.

To illustrate this more fully, consider the finding of the instantaneous center when two positions of the moving body are given instead of the velocities of any two of its points. In Fig. 29, let A_1B_1 and A_2B_2 be the initial and final positions of the body; erect perpendicular bi-

sectors MO and NO to A_1A_2 and B_1B_2; these will meet at the instantaneous center, O. To prove this it must be shown that $\angle A_1OA_2 = \angle B_1OB_2$, as all points of a body in rotation must describe equal angles in equal times about the center of rotation.

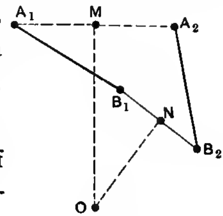


FIG. 29

EXERCISE 113. Give a geometric proof of the correctness of the above construction.

If now several positions of the moving body be known, an instantaneous center may be found for each successive displacement. This is illustrated in Fig. 30, where the

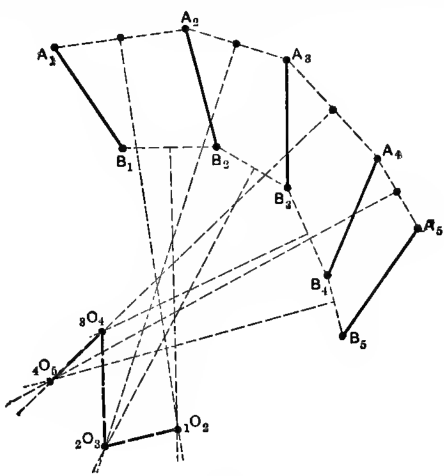


FIG. 30

successive instantaneous centers are denoted by ${}_1O_2$, ${}_2O_3$, ... The line joining these instantaneous centers will be the centrode of the body, two of whose points are A and B.

The closer the successive positions of the body are taken the more accurately can the centrode be plotted. In order to fully imitate the actual motion of the body, its successive positions must be taken indefinitely near to one another and thus the number of instantaneous centers will be indefinitely increased.

In the case of a rolling wheel the instantaneous axis is the point of contact with the ground and the centrode is the path traced by the wheel upon the ground. The motion of the wheel may also be considered as a rotation about its axle as axis combined with a translation parallel to the ground.

EXERCISE 114. A ladder slides between a vertical wall and horizontal ground. Find the instantaneous axis for several positions of the ladder. Find the centrode.

EXERCISE 115. Considering the motion of the ladder at any instant as a rotation about its instantaneous center, find the direction of the motion of any point in the ladder.

EXERCISE 116. A wheel 8 feet in diameter rolls along the ground. Find the height above the ground of a point in the circumference moving at half the velocity of the highest point of the wheel.

EXERCISE 117. Find the instantaneous center for several positions of the connecting-rod of an engine whose stroke is 2 feet and whose connecting-rod is 4 feet long.

EXERCISE 118. If the velocity of the cross-head of the engine of Exercise 117 is 10 ft. per sec. and the crank makes an angle of 60° with the horizontal find the velocity of the crank-pin.

KINETICS

CHAPTER IV

KINETICS

Section X

INTRODUCTION

Having studied motion in the abstract under the head of Kinematics, due regard must now also be given to the mass of the body moved and the force producing the motion. This part of Mechanics is known as *Kinetics*.

In the study of Statics we were introduced to the term *Stress*. When one portion of matter acts on another portion, then the whole phenomenon of the mutual action of the two portions of matter is called a stress. Stresses sometimes receive special names, such as tension, compression, torsion, attraction, or repulsion.

It is often useful to concentrate our attention upon the one or the other of the portions of interacting matter and thus consider force as one aspect of the stress. We then speak of the action of *External* or *Impressed Force* upon the portion of matter under consideration, and we say that this force is due to the *action* of the other portion of matter. The opposite aspect of the same stress is then called the *reaction* on the other portion of matter.

Thus the forces denoted as *action* and *reaction* are simply different views of the same stress. This may be

likened to the different aspects of the same transaction expressed by the correlative terms of buying and selling.

The relations of matter, motion, and force which constitute that portion of Mechanics known as Kinetics may be based upon three postulates, known as Newton's Laws of Motion. These laws were known to Galileo and other predecessors of Newton, but were first stated in concise terms by Newton.

No direct proof of Newton's Laws is possible. The correctness of the laws is established by assuming the laws true and then testing, by experiment or observation, the deductions to which they lead. This has been done in so many ways, by so many observers, that their acceptance as true is justified.

This has been particularly the case in astronomy. For example, the Nautical Almanac, or Ephemeris, containing the positions of the planets, etc., for each day and hour, is published years beforehand, and these predictions, based upon Newton's Laws, are always found to agree with the occurrence when observed.

Newton's Laws of Motion deal with the effect of external or impressed forces on the motion of bodies.

The first law deals with the behavior of a body when no external forces act on it.

The second law tells how the external force, when acting, may be measured.

The third compares the two aspects of stress, i.e., action and reaction.

Newton's First Law. "Every body continues in its state of rest or uniform motion in a straight line, unless it be compelled by impressed forces to change that state."

This law is known as the law of *Inertia*, for it states

that a body cannot of itself change its condition of rest or motion. It may be stated by saying that matter is *inert.*

This law gives us a definition of force, for it says that without the action of a force there can be no change of motion.

It also tells us how a body will move if no external forces act upon it.

Newton's Second Law. "Change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force acts."

This law gives us the means of measuring forces and masses and calculating their effects.

To fully understand this law it must be remembered that by the term "change of motion" Newton meant what we now call change of momentum and by "impressed force" Newton meant the impulse of the impressed force.

Momentum, or, as it is sometimes called, quantity of motion, is defined as the product of the mass of the moving body and its velocity.

The *Impulse* of a force is defined as the product of the force and the time during which it acts.

Therefore Newton's Second Law may be expressed algebraically thus:

$$Ft \propto m(v_2 - v_1),$$

where F represents the external force acting, t the time during which it changes the velocity of the mass, m, from v_1 to v_2.

If c be the *proportionality factor,* we have

$$Ft = cm(v_2 - v_1),$$

also $$F = cm\frac{v_2 - v_1}{t},$$

where $\frac{v_2 - v_1}{t}$ is the acceleration produced by the force F which we will denote by a.

$$\therefore\ F = cma.$$

In engineering practice the unit of force is always taken as the force with which the earth attracts a certain lump of platinum (marked "P.S., 1844, 1 lb.", and carefully preserved in London). This force is called a pound and this force is our unit of force.

The unit of acceleration has always been fixed as an acceleration of 1 ft.-per-sec. per sec.

The units of force and acceleration having been thus fixed it remains to determine the unit of mass.

To do this consider a freely falling body of mass m and weight W. Its acceleration is $g(=32.19$ at London) ft.-per-sec. per sec., as is determined experimentally. The only external force acting upon the mass m to produce this acceleration g is its weight of W pounds.

Therefore adapting the formula $F = cma$ to this case we have

$$W = cmg \quad \text{or} \quad m = \frac{W}{cg}.$$

If now m is to be a unit of mass and c is to be unity (for it is naturally most convenient to so choose the unit of mass as to make the proportionality factor, c, unity), then the equation $m = \frac{W}{cg}$ can only hold if W, the weight,

be numerically equal to g pounds. for only then is $1=\frac{g}{(1)(g)}=1$.

This means that the unit of mass must be a quantity of matter which weighs $g(=32.19)$ pounds. This unit of mass is called the *gravitational unit of mass*, sometimes also the *engineer's unit of mass.*

Thus a lump of iron weighing 10 g pounds is attracted towards the earth with a force of 10 g pounds and contains $\frac{10\,g}{g}=10$ gravitational units of mass.

EXERCISE 119. What is the mass of 100 lbs. of iron and what does the 100 lbs. mean?

The fundamental formula of Kinetics is

$$\mathbf{F=ma.}$$

It is a direct consequence of the second law of motion. It connects the acceleration, a, measured *in ft.-per-sec.* per sec., communicated to a mass, m, *measured in gravitational units of mass*, with the external force, F, *measured in pounds*, which acts upon the mass.

Example.—How long must a force of 6 pounds act on a mass of 96 pounds in order to generate a velocity of 40 ft. per sec.? ($g=32$.)

Solution.—Use the formula $F=ma$. We can find the mass, m, of the body for a body weighing 96 pounds contains $\frac{96}{32}=3$ units of mass.

The force, F, is 6 pounds.

$$\therefore\ 6=3a, \quad \therefore\ a=2 \text{ ft.-per-sec. per sec.}$$

The time necessary to generate a velocity of 40 feet per sec. starting from rest with an acceleration of 2 ft.-per-sec. per sec. may now be found by the kinematical formula $v=at+v_0$; thus, $40=2(t)+0$ or $t=20$ seconds. This is the time during which the force of 6 pounds must act.

EXERCISE 120. A force of 4 pounds causes a certain mass to move from rest through 18 feet in 3 seconds; find the mass. How much does this mass weigh?

EXERCISE 121. A mass of 64 pounds originally at rest is acted on by a constant force which acts for 5 seconds and then ceases to act; the body moves through 60 feet in the next 2 seconds. Find the force.

EXERCISE 122. A mass of 200 pounds is acted on by a force of 10 pounds for 20 seconds. What distance will the mass have passed over and what velocity will it have acquired if its initial velocity was 5 ft. per sec.?

EXERCISE 123. In what time will a force of 5 pounds move a mass of 16 pounds through 45 feet along a smooth horizontal plane?

EXERCISE 124. What will be the time in Ex. 123 if the plane be rough and the coefficient of friction is 0.25?

EXERCISE 125. What force must act on a mass of 48 pounds to increase its velocity from 30 to 40 ft. per sec. while it passes over 80 feet?

EXERCISE 126. What is the friction force if a body weighing 20 pounds and projected along a rough horizontal plane with a velocity of 48 ft. per sec. comes to rest after 5 seconds?

Newton's Third Law.—"To every action there is always an equal and contrary reaction, or the mutual actions of any two bodies are always equal and oppositely directed."

This law brings to our attention the fact that a force

is simply one aspect of the stress existing between two portions of matter.

EXERCISE 127. Explain how this law is applied to the case of a horse towing a boat.

Is the following statement correct?

"The forward pull exerted by the horse on the tow-rope is exactly equal to the backward pull exerted by the tow-rope on the horse."

SECTION XI

DIGRESSION AS TO THE THEORY OF DIMENSIONS

Every physical quantity has a definite magnitude. This magnitude we may not in all cases be able to measure with great accuracy, yet the definiteness of the magnitude of the quantity remains.

To measure a physical quantity we employ a certain fixed amount of the *same* physical quantity which we call the unit of *that particular quantity*. The given quantity is then said to be equal to so many times this unit.

Thus, to measure a certain *length* we must first decide upon a *certain fixed length*, say the foot, and then determine *how many times this length* is contained in the given length. If the unit length is contained x times in the given length, we say the given length is x *units, feet.*

It is thus seen that the result of the measurement of a physical quantity must consist of two parts: first, a pure number, which states the number of times the unit is contained in the given quantity, and second the name of the unit which has been employed.

When we write the formula $v=\frac{s}{t}$ we mean not only

that the number of units of velocity is the number of units of space divided by the number of units of time, but there underlies this mode of expression a tacit understanding that we adhere consistently to some definite system of units.

So that the formula might more completely be written

$$v \text{ (ft. per sec.)} = \frac{s \text{ (feet)}}{t \text{ (seconds)}}.$$

This is usually abbreviated to

$$v[V] = \frac{s[S]}{t[T]},$$

where the v, s, and t represent numbers and bracketed capitals the conventional units only.

If v, s, and t all become equal to 1, then we have

$$[V] = \frac{[S]}{[T]}.$$

This equation is technically known as an *Equation of Dimensions.*

The units used in Mechanics are of two kinds, fundamental and derived. It is usual to assume as fundamental the units of length, mass, and time, denoted by $[L]$, $[M]$, and $[T]$, and to consider all other units as derived from these.

The relation by means of which we derive the magnitude of the unit of any quantity in terms of the fundamental units is indicated by what are called the *dimensions* of the unit in question.

Thus consider the unit of velocity. Its dimensional equation has been found to be

$$[V]=\frac{[S]}{[T]} \quad \text{or} \quad =\frac{[L]}{[T]};$$

$$\therefore\ [V]=[LT^{-1}] \quad \text{or} \quad [LM^0T^{-1}].$$

We therefore say that the dimensions of velocity are the first dimension in length, zero dimension in mass, and the minus first dimension in time, or, more concisely, $[L^1M^0T^{-1}]$ gives the dimensions of velocity.

Again if we indicate the unit of area by the symbol A, we have $[A]=[LL]=[L^2]$, since the unit of area is a square of which the sides are each of unit length, and since area has no connection with mass or time, we may state $[A]=[L^2M^0T^0]$, which by the laws of Algebra reduces to $[A]=[L^2]$.

Similarly the dimensions of volume are $[L^3M^0T^0]$.

EXERCISE 128. Deduce the dimensions of acceleration, momentum, impulse, and force from their defining equations.

From the definition of an angle $\theta=\frac{s}{r}$, page 53, we may deduce its dimensions as follows:

$$\theta[\theta]=\frac{s[L]}{r[L]}.$$

Putting θ, s, and r each equal to unity, we have

$$[\theta]=\frac{[L]}{[L]}=[L^0]=[L^0M^0T^0].$$

This shows that an angle is a mere number and has no dimensions.

EXERCISE 129. Find the dimensions of the trigonometric functions.

EXERCISE 130. Find the dimensions of angular velocity, angular acceleration, normal acceleration.

Dimensional Equations are of use in two ways:

(1) They afford a check on the accuracy of the line of reasoning by means of which we have deduced an equation connecting any physical quantities. For since it is impossible to compare physical quantities which are not of the same kind, it follows that the dimensions of the two sides of any equation connecting physical quantities must be the same.

(2) They afford a means by which we can convert the magnitude of any physical quantity expressed in terms of the units belonging to one system into those of any other system.

The first use is of great importance and should always be used to check any equation between physical quantities which we may deduce. This is illustrated by the following:

Example.—Suppose we had in some manner reached the conclusion that the volume of water, c, which passes any point of a stream during a time, t, was given in terms of the cross-section of the stream, a, and its velocity, v. by the equation

$$c = va^2t.$$

Then expressing the units in full, we have

$$c[C] = v[V]a^2[A^2]t[T],$$

and putting c, v, a, and t equal to unity, we have

$$[C] = [V][A^2][T],$$

or in terms of the fundamental units,

$$[L^3]=[LT^{-1}][L^4][T]=[L^5].$$

Here it is seen that the dimensions of the quantities on both sides of the equation $c=va^2t$ are not the same, therefore the equation is incorrect.

In fact the true equation is $c=vat$.

EXERCISE 131. Test the last statement by the theory of dimensions.

In applying the theory of dimensions to equations it must be remembered that only *quantities of the same kind can be added or subtracted and that the result is a quantity of the same kind*, so that

$$[L^3]+[L^3]=[L^3] \quad \text{and not} \quad 2[L^3],$$

as it may seem by Algebra, for the statement is equivalent to

["cubic feet"]+["cubic feet"] give ["cubic feet"].

Also if logarithms are involved in an equation it must be remembered that only the logarithm of a *number* can be taken so that the quantity whose logarithm is to be taken must have no dimensions.

EXERCISE 132. Check the result of the Example on page 15, both in the exponential and logarithmic form.

EXERCISE 133. Check by the theory of dimensions the equations

$$\theta=2\pi nt,$$
$$v=r\omega\cos(\omega t+\phi),$$
$$a=-r\omega^2\sin(\omega t+\phi),$$

deduced in Section II.

EXERCISE 134. Check result of Exercise 55.

EXERCISE 135. Check equations (1), page 23, and equations (2), page 56.

The second use of dimensional equations is illustrated by the following:

Example.—Find the ratio between the F.P.S. and the C.G.S. unit of velocity.

The former is 1 ft. per sec., the latter is 1 cm. per sec., and 1 foot = 30.48 cm.

$$\therefore \text{ as } [V] = \left[\frac{L}{T}\right] = \left[\frac{\text{foot}}{\text{second}}\right]$$

$$= \left[\frac{30.48 \text{ cm.}}{\text{second}}\right] = 30.48\left[\frac{\text{cm.}}{\text{second}}\right].$$

$\therefore$ the F.P.S. unit is 30.48 times the C.G.S. unit.

EXERCISE 136. How many F.P.S. units of force (poundals) is a C.G.S. unit of force (dyne) equal to? Assume 1 gram = 15.432 grains, 7000 grains = 1 pound, 1 foot = 30.48 cm.

CHAPTER V

KINETICS OF A PARTICLE AND OF THE MASS-CENTER OF A RIGID BODY

SECTION XII

EQUATIONS OF MOTION FOR TRANSLATION

IN Section X it was shown that a particle acted on by a single force F obtains thereby an acceleration, a, in the direction in which the force F acts, and that if m be the mass of the particle, the formula $F = ma$ connects these quantities.

If the particle be acted on by several forces the resulting acceleration may still be found by means of the formula $F = ma$, provided F now stands for the resultant of the forces acting upon the particle, found by means of the principles of "Statics".

EXERCISE 137. Find the acceleration, and its direction, of a particle weighing 64 pounds which is acted on by forces of 10, 20, and 30 pounds directed towards the east, northeast, and 30° south-of-west, respectively.

Equations of Motion of a Particle

In Fig. 31, let F', F'', ... be forces acting on a particle whose mass is m; also let R be their resultant and θ its inclination to the x-axis.

Then $R = ma$, where a is the acceleration produced by the forces F', F'', . . . , and a has the same direction as R.

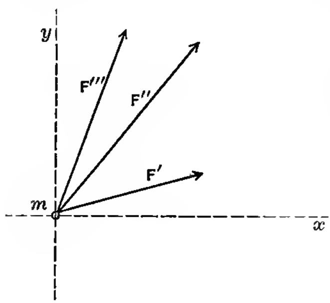


FIG. 31

Multiply this equation by $\cos\theta$ and $\sin\theta$, respectively, then $R\cos\theta = ma\cos\theta$ and $R\sin\theta = ma\sin\theta$. But $R\cos\theta = \Sigma F_x$ and $R\sin\theta = \Sigma F_y$ (by the principles of "Statics") and $a\cos\theta = a_x$ and $a\sin\theta = a_y$, where a_x and a_y are the axial components of the resulting acceleration;

$$\therefore\ \Sigma \mathbf{F}_x = \mathbf{m}\mathbf{a}_x \quad \text{and} \quad \Sigma \mathbf{F}_y = \mathbf{m}\mathbf{a}_y.$$

These are the equations of motion of a particle.

EXERCISE 138. Find a_x and a_y due to the forces in Exercise 137, and thus find a.

Mass-center of a System of Particles

Consider now a *system of particles.* If the distances between the particles of the system remain invariable, the system is said to be rigid and the system of particles constitutes a rigid body.

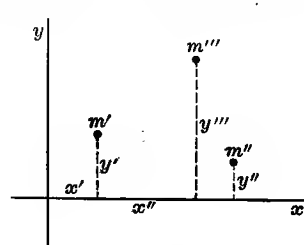


FIG. 32

In Fig. 32 consider the rigid system composed of the particles m', m'', m''', . . .

Then $m'x'$, . . . and $m'y'$, . . . are called the moments

of the masses of the particles about the respective axes and the point $(\bar{x}, \bar{y})$, where

$$\bar{\mathbf{x}}=\frac{\Sigma \mathbf{mx}}{\Sigma \mathbf{m}} \quad \text{and} \quad \bar{\mathbf{y}}=\frac{\Sigma \mathbf{my}}{\Sigma \mathbf{m}}$$

is called the *mass-center of the body.*

As $$m=\frac{W}{g},$$

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}=\frac{\Sigma \frac{W}{g} x}{\Sigma \frac{W}{g}}=\frac{\Sigma Wx}{\Sigma W},$$

and similarly $$\bar{y}=\frac{\Sigma Wy}{\Sigma W};$$

$\therefore$ *the mass-center coincides with the centroid,* already studied in "Statics".

Equations of Motion of the Mass-center of a System of Particles

Among the forces acting on the system of particles, beside those exerted by the particles upon themselves, some may be exerted by particles not belonging to the system. These are called *external forces* to distinguish them from the *internal forces* exerted by the particles of the system upon themselves.

According to the third law of motion, if one particle exerts a force upon another, the second exerts an equal and opposite force upon the first. The *internal forces*

thus always occur in pairs in the form of stresses and *cannot affect the motion of the system as a whole.*

Thus, in considering the motion of a system as a whole, only the external forces acting upon it need be considered.

Let the x-components of the accelerations (due to the external forces) of each mass of the system, Fig. 32, be a_x', a_x'', . . . , and let a_x be the component acceleration of the *mass-center*, then

$$m'a_x' + m''a_x'' = a_x \Sigma m,$$

for by the definition of *mass-center* we have

$$m'x' + m''x'' \ldots = \bar{x}\Sigma m,$$

by differentiation

$$m'\frac{dx'}{dt} + m''\frac{dx''}{dt} + \ldots = \frac{d\bar{x}}{dt}\Sigma m,$$

and $$m'\frac{d^2x'}{dt^2} + m''\frac{d^2x''}{dt^2} + \ldots = \frac{d^2\bar{x}}{dt^2}\Sigma m,$$

which is equivalent to $m'a_x' + m''a_x'' + \ldots = a_x\Sigma m$.

Now, if F_x', F_x'', . . . represent the x-components of the external forces acting upon the particles m', m'', . . . , respectively, then

$$m'a_x' = F_x', \quad m''a_x'' = F_x'', \ldots,$$

and thus

$$F_x' + F_x'' + \ldots = a_x\Sigma m,$$

or $$\Sigma \mathbf{F}_x = \mathbf{a}_x \Sigma \mathbf{m},$$

and similarly, $$\Sigma \mathbf{F}_y = \mathbf{a}_y \Sigma \mathbf{m},$$

or the *acceleration of the mass-center of a system of particles is the same as that of a particle whose mass equals that of the system and which is acted on by forces equal and parallel to the external forces applied to the system.*

The application of these equations of motion can best be understood by some examples.

Example.—A body rests upon a smooth horizontal surface, its weight is 64 pounds, and a force of 50 pounds inclined at 60° to horizon acts upon it. Find the acceleration and the reaction of the support.

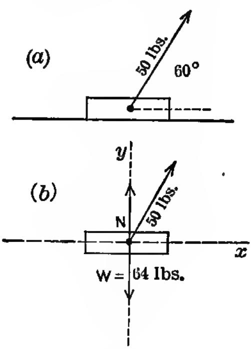


FIG. 33

Solution.—Fig. 33 (*a*) illustrates the problem. In Fig. 33 (*b*) the body is shown as a "*free body*" (*with all external forces acting upon it*). Evidently the only acceleration will be horizontal. Therefore select the position of the axes as shown.

From $$\Sigma F_x = a_x \Sigma m,$$

and $$\Sigma F_y = a_y \Sigma m,$$

we have $$50 \cos 60° = a_x(\tfrac{64}{32}),$$

and $$N + 50 \sin 60° - 64 = a_y(\tfrac{64}{32}).$$

$$\therefore\ a_x = \tfrac{25}{2} \text{ ft.-per-sec. per sec.}$$

As $$a_y = 0,$$

$$N = 64 - 50\frac{\sqrt{3}}{2} = 64 - 25\sqrt{3} \text{ lbs.}$$

Example.—In an elevator are two boxes weighing 640 and 800 pounds, the lighter box being on top of the other. What is the pressure on the bottom of each box if the elevator is started up with an acceleration of 4 ft.-per-sec. per sec.? How much of this pressure is due to the accelerated motion?

Solution.—Fig. 34 (*a*) illustrates the problem. Fig. 34 (*b*) shows the lighter box as a *free body*, where P is the upward pressure exerted on the box by the lower one *when in motion.*

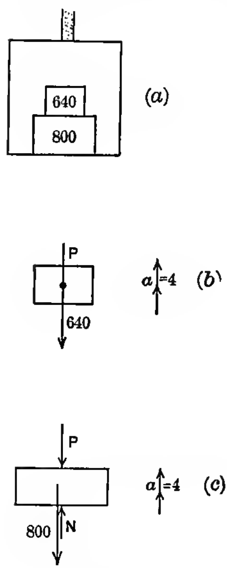


FIG. 34

Thus as

$$\Sigma F_y = a_y \Sigma m,$$

$$-640 + P = (4)(\tfrac{640}{32}),$$

$$P = 640 + 80 = 720 \text{ lbs.}$$

Again from Fig. 34 (*c*),

$$-800 - P + N = 4(\tfrac{800}{32}),$$

$$N = 100 + 800 + P$$

$$= 1620 \text{ lbs.}$$

If an elevator were at rest it is evident that the pressure would be 640 on the bottom of the upper box and 1440 on the bottom of the lower box, therefore the pressures due to accelerated motion are $720 - 640 = 80$ lbs., and $1620 - 1440 = 180$ lbs., respectively.

EXERCISE 139. What is the tension in the cable supporting the elevator in the above example if the weight of the elevator

is neglected: (*a*) when the elevator is at rest, (*b*) when the elevator is in accelerated motion $a = -2$ ft.-per-sec. per sec., (*c*) when the elevator is moving with a constant velocity of 10 ft. per sec.?

EXERCISE 140. Same as Exercise 139 if the weight of the elevator is 1500 pounds (assume the system considered to be composed of the elevator plus the two boxes).

EXERCISE 141. If the weight of the air displaced by a balloon (which is the upward force) is 4800 pounds and the weight of the balloon is 4500 pounds, with what acceleration does it begin to ascend? How long will it take it to ascend to a height of 200 feet, assuming it to continue with the starting acceleration?

EXERCISE 142. An ice-boat weighing 1000 lbs. is driven for one minute from rest by a wind force of 100 pounds. Find the velocity acquired and the distance passed over if the coefficient of friction is 0.02.

EXERCISE 143. A train of 100 tons is running at the rate of 45 miles an hour. Find the constant force necessary to stop it in one minute. How far will the train move before coming to rest?

EXERCISE 144. What pressure will a man weighing 150 pounds exert on the floor of an elevator ascending with an acceleration of (*a*) 10 ft.-per-sec. per sec., (*b*) 32 ft.-per-sec. per sec.?

EXERCISE 145. Same as Exercise 144, but the elevator descending with an acceleration of (*a*) 10 ft.-per-sec. per sec.
(*b*) 32 " " "
(*c*) 40 " " "

EXERCISE 146. If the elevator in Exercise 144 descends with a constant velocity for 100 feet, what will be the pressure exerted?

APPLICATION OF THE EQUATIONS OF MOTION FOR TRANSLATION

Section XIII

TRANSLATION DUE TO CONSTANT FORCES

Motion on Inclined Planes

Example.—A body is projected up a smooth inclined plane with a velocity of 40 feet per second. If the inclination of the plane is 30°, find the distance that the body will move up the plane before coming to rest.

Solution.—Fig. 35 (*a*) illustrates the problem; Fig. 35 (*b*) shows the body as a "free body".

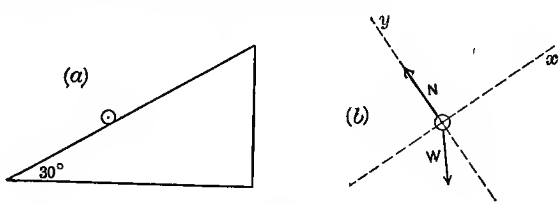


Fig. 35

As the direction of motion is parallel to the plane take the x- and y-axes as shown. As a_x is the only acceleration, we need only use the equation $\Sigma F_x = a_x \Sigma m$.

Thus, $-W \sin 30 = a_x m$, but $m = \dfrac{W}{g}$;

$$\therefore \quad -\frac{W}{2} = a_x \frac{W}{g} \quad \text{or} \quad a_x = -\frac{g}{2}.$$

This acceleration is minus, as we assume the direction up the plane as plus.

To find the required distance use the kinematical formula

$$v^2=2as+v_0^2, \quad \therefore\ 0=2\left(-\frac{g}{2}\right)s+(40)^2,$$

and $$s=\frac{1600}{g}=50 \text{ feet.}$$

EXERCISE 147. A body is projected down a smooth inclined plane whose height is $\frac{1}{10}$ of its length with a velocity of $7\frac{1}{2}$ miles per hour. How far will it travel in 2 minutes?

EXERCISE 148. A body weighing 30 pounds falls down a rough inclined plane, height 30 feet and base 100 feet. If $\mu=\frac{1}{5}$, what is the velocity acquired?

EXERCISE 149. Show that the time of falling down any smooth chord drawn from the highest point of a vertical circle is the same as the time of falling through the vertical diameter. (Assume any chord inclined at θ to the vertical diameter and show that the time is independent of θ.)

EXERCISE 150. A trolley car at the top of a hill becomes uncontrollable and "runs wild" down a grade of 1 vertical to 20 horizontal a distance of $\frac{1}{4}$ mile. The resistance due to friction, etc., being 20 pounds per ton and the weight of car and passengers 50 tons, how fast will the car be moving when it reaches the foot of the hill?

EXERCISE 151. A train of 100 tons, excluding the engine, runs up a 2% grade with an acceleration of 1 ft.-per-sec. per sec. If the friction is 10 pounds per ton, find the pull on the draw-bar between the engine and train.

Projectiles in Vacuo

Consider a ball projected into the air where resistance is neglected, or into a vacuum.

Let the initial velocity of the mass-center of the ball

be v_0, its inclination to the horizontal θ, and the time of projection be $t=0$.

Consider the ball as a free body at any subsequent time, as $t=t$. The only force then acting on the ball will be its weight W pounds, and the *equations of motion*

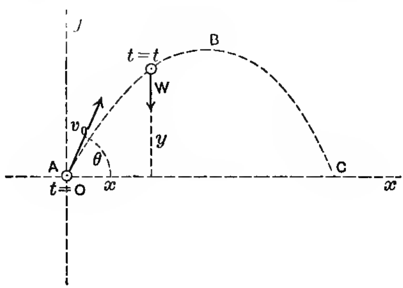


FIG. 36

of its mass-center referred to the axes shown in Fig. 36 will then be

$$0=ma_x \quad \text{and} \quad -W=ma_y, \quad . \quad . \quad . \quad (1)$$

where m is the mass of the ball and equals $\dfrac{W}{g}$ units of mass.

As x and y are the coordinates of the position of the mass-center of the ball at the time t, we have

$$a_x=\frac{d^2x}{dt^2} \quad \text{and} \quad a_y=\frac{d^2y}{dt^2}.$$

Therefore by equations (1)

$$\frac{d^2x}{dt^2}=0 \quad \text{and} \quad \frac{d^2y}{dt^2}=-g.$$

Integrating these equations and determining the constants of integration by means of the initial conditions of the motion, we have

$$v_x=\frac{dx}{dt}=v_0 \cos \theta \quad \text{and} \quad v_y=\frac{dy}{dt}=v_0 \sin \theta - gt, \quad (2)$$

which are the component velocities of the mass-center of the ball at any instant.

EXERCISE 152. Complete the above indicated steps. Are these component velocities constant? Deduce the actual velocity at any instant. What is its direction?

Integrating the above equations again and introducing the values of the new constants of integration, we have

$$x=v_0 \cos \theta \cdot t \quad \text{and} \quad y=v_0 \sin \theta \cdot t - \tfrac{1}{2}gt^2. \quad . \quad (3)$$

These are the coordinates of the mass-center of the ball at any time t in terms of the constants of the problem and the variable time.

They are the parametric equations of the path of the projectile.

EXERCISE 153. Deduce the rectangular equation of the path of the ball from its parametric equations.

The above equations give us the *equation of the trajectory*

$$y=x \tan \theta - \frac{gx^2}{2v_0^2 \cos^2 \theta}. \quad . \quad . \quad . \quad . \quad (4)$$

The *horizontal range* of the projectile is the distance AC (Fig. 36) and the time required to reach C from A is called the *time of flight.*

EXERCISE 154. From equation (4) deduce the range $\left(R=\frac{v_0^2 \sin 2\theta}{g}\right)$ by putting $y=0$ and thus finding the "intercepts on the X-axis".

EXERCISE 155. What value of θ in $R=\frac{v_0^2 \sin 2\theta}{g}$ gives the *maximum range?*

EXERCISE 156. By the Calculus find the maximum value of y from Equation (4). This gives the *greatest height* $H=\frac{v_0^2 \sin^2 \theta}{2g}$.

EXERCISE 157. From equation (3), knowing the abscissa of B (Fig. 36), the highest point reached by the projectile, to be $\frac{1}{2}R=\frac{v_0^2 \sin \theta \cos \theta}{g}$, find the time of flight. $\left(T=\frac{2v_0 \sin \theta}{g}\right)$.

EXERCISE 158. In Analytic Geometry it is shown that if an equation is in the form

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0,$$

the curve is a conic section and its nature is determined by B^2-4AC. If this quantity is <0, $=0$, or >0 the curve is an ellipse, parabola, or hyperbola respectively. Examine eq. (4) by this method. What kind of curve is the path of a projectile in vacuo?

It is shown that the trajectory in vacuo is a parabola. In air the path of any small dense object (a ball of metal)

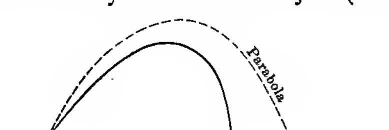


FIG. 37

having a moderate initial velocity approximates closely the parabolic path. But the course of a cannon-ball

moving with great initial velocity is much affected by the resistance of the air and its trajectory compares to the parabolic path somewhat as shown in Fig. 37.

The above discussion shows

(*a*) that the horizontal velocity of a projectile is constant, for from equation (2)

$$v_x = v_0 \cos \theta;$$

(*b*) that the vertical motion of a projectile may be treated as a case of vertical projection, for the formulæ

$$v_y = v_0 \sin \theta - gt,$$

$$y = v_0 \sin \theta \cdot t - \tfrac{1}{2}gt^2$$

correspond to the formulæ

$$v = at + v_0,$$

$$s = \tfrac{1}{2}at^2 + v_0 t,$$

already deduced from any motion under constant acceleration, when $a = -g$ and $v_0 = v_0 \sin \theta$;

(*c*) that the horizontal range is $\dfrac{2v_0^2 \sin \theta \cos \theta}{g}$ = (horizontal component of the initial velocity) (time of flight).

Example.—A body projected at an inclination of 45° to the horizon from the top of a tower fell in 5 seconds at a distance from the foot equal to the height of the tower. Find the height of the tower and the initial velocity.

Here the time of flight is 5 sec. Let the initial velocity be v feet per sec. and the height of the tower x feet.

Then Fig. 38 illustrates the problem.

Decompose the initial velocity into vertical and horizontal components, i.e., $v \sin 45 = v\frac{\sqrt{2}}{2}$ and $v \cos 45 = v\frac{\sqrt{2}}{2}$.

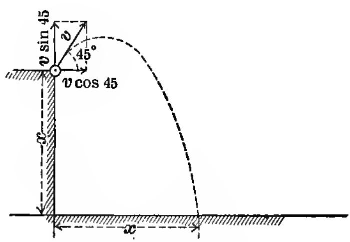


FIG. 38

Then as Range = (horizontal component of the initial velocity) (time of flight), we have

$$x = \left(v\frac{\sqrt{2}}{2}\right)(5) = \frac{5v\sqrt{2}}{2}.$$

Also, as $$s = \tfrac{1}{2}at^2 + v_0t,$$

$$-x = \tfrac{1}{2}(-32)(25) + \left(v\frac{\sqrt{2}}{2}\right)(5).$$

Solving these equations for v and x we have

$$x = 200 \text{ feet} \quad \text{and} \quad v = 40\sqrt{2} \text{ ft. per sec.}$$

EXERCISE 159. A body is projected with a velocity of 20 ft. per sec. at an elevation of 45°. Find its greatest height and the horizontal range.

EXERCISE 160. A 20-pound weight is dropped from a window of a car traveling over a bridge at the rate of 30 miles an hour. How long will it take to reach the water 75 feet below?

How much in advance of its position when dropped will it strike the water?

EXERCISE 161. A body projected at an angle of 60° to the horizon with a velocity of 40 ft. per sec. strikes the perpendicular face of a tower at a horizontal distance of 20 feet from the point of projection. Find the height at which it strikes the tower.

EXERCISE 162. A piece of ice is detached from a roof, whose slope is 30°, at a point 8 feet from the eaves, which are 24 feet above the ground. At what distance from the vertical plane through the eaves will it reach the ground? (Neglect all friction.)

EXERCISE 163. At what angles of projection is the horizontal range equal to the height due to the initial velocity?

EXERCISE 164. A nozzle delivering water at constant velocity can be moved to any angle in a fixed vertical plane. Find the equation of the curve bounding the area which may be reached by the water, assuming the parabolic path.

(*Hint.*—In equation (4) consider θ the variable parameter and find the envelope of the resulting family of parabolas.)

Example.—A ball fired with a velocity u at an inclination α to the horizon just clears a vertical wall which subtends an angle, β, at the point of projection. Determine the instant at which the ball just clears the wall.

Solution.—Fig. 39 illustrates the problem. To solve the problem find the equation of the trajectory and the equation of the line OB. Their intersection will give the abscissa of the top of the wall B. By dividing this distance by the horizontal component of the initial velocity we obtain the time required to reach

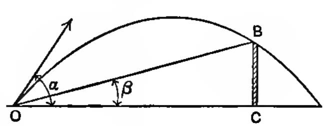


FIG. 39

B from O. Thus, the equations of motion of the projectile are

$$0 = ma_x \quad \text{and} \quad -w = ma_y,$$

or
$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

By integrating and determining the constants of integration we have

$$v_x = u \cos \alpha \quad \text{and} \quad v_y = -gt + u \sin \alpha.$$

Integrating again we have

$$x = u \cos \alpha t \quad \text{and} \quad y = -\frac{gt^2}{2} + u \sin \alpha t,$$

and the equation of the trajectory is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}. \quad . \quad . \quad . \quad . \quad (1)$$

The equation of the line OB is

$$y = x \tan \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving (1) and (2)

$$x = \frac{2u^2 \cos^2 \alpha (\tan \alpha - \tan \beta)}{g} = \frac{2u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos \beta};$$

this is the abscissa, OC, of the point B.

As the horizontal component of the velocity of projection is constant and equal to $v_x = u \cos \alpha$ the time on reaching B will be

$$\frac{OC}{v_x} = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$$

seconds after leaving O.

EXERCISE 165. In the preceding example find the horizontal distance between the foot of the wall and the point where the ball strikes the ground.

EXERCISE 166. At the distance of a quarter of a mile from the bottom of a cliff, which is 120 feet high, a shot is to be fired which shall just clear the cliff and pass over it horizontally; find the angle of projection and the velocity of projection.

EXERCISE 167. A stone thrown with a velocity of 64 ft. per sec. is to hit an object on the top of a wall 19 feet high and 48 feet distant. Determine the angle of projection.

SECTION XIV

TRANSLATION DUE TO VARIABLE FORCES

So far we have only considered the application of the equations of motion to phenomena involving constant forces which produced constant accelerations in the masses acted on.

The equations of motion are, however, always applicable. When the forces involved are variable the acceleration produced is necessarily variable and should then always be represented by $\frac{dv}{dt}$, $\frac{d^2s}{dt^2}$, or $v\frac{dv}{ds}$.

As an *example* let us consider the motion of a particle P of mass m (Fig. 40) acted on by a force F always directed towards a fixed point C and proportional to the distance of P from C.

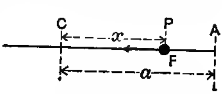


FIG. 40

Here the motion evidently takes place in the straight line joining P and C. Let x be the distance from P to C; then the acceleration of P may be put equal to $\frac{dv}{dt}$,

$\frac{d^2x}{dt^2}$, or $v\frac{dv}{dx}$, wherein v is the velocity of P at a distance x from C and t is the corresponding time.

As F is proportional to x, we may put $F = -cx$, where c is the proportionality factor and the sign is negative as the force is directed to the left. The formula $F = ma$ thus becomes

$$-cx = mv\frac{dv}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Why is this particular expression for the acceleration selected?) Equation (1) must now be integrated and solved for v.

Thus $-cx\,dx = mv\,dv$,

$$\therefore \quad -\frac{cx^2}{2} = m\frac{v^2}{2} + C_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C_1 is the constant of integration. To determine this constant, we must return to the physical conditions of the problem. We will suppose that P started from rest at a distance a to the right of C; thus, note that when $x = a$, $v = 0$. Substituting these values in equation (2) we have

$$-\frac{ca^2}{2} = m\frac{(0)^2}{2} + C_1,$$

$$\therefore \; C_1 = -\frac{ca^2}{2};$$

so that (2) becomes $-\frac{cx^2}{2} = m\frac{v^2}{2} - \frac{ca^2}{2}$.

$$\therefore \; v^2 = \frac{c}{m}(a^2 - x^2) \quad \text{or} \quad v = \sqrt{\frac{c}{m}}\sqrt{a^2 - x^2}. \quad . \quad (3)$$

Consider the velocity of P as determined by equation (3) for various values of x.

Make $x>a$;	then v has an imaginary value (does not exist).
" $x=a$;	" $v=0$.
" $x<a$ but >0;	" v has either a positive or negative value.
" $x=0$;	" $v=\pm a\sqrt{\frac{c}{m}}$.
" $x<0$ but $>-a$;	" v has either a positive or negative value.
$x=-a$;	" $v=0$.
" $x<-a$;	" v has an imaginary value.

From this we learn that P has an oscillatory motion of amplitude a with C as central position. From equation (1) we note that $v\frac{dv}{dx}=-\frac{c}{m}x$, or the acceleration is proportional to the displacement, and therefore the particle possesses simple harmonic motion (see page 29).

From equation (3) by substituitng $v=\frac{dx}{dt}$ and integrating we obtain x as follows:

$$\frac{dx}{dt}=\sqrt{\frac{c}{m}}\sqrt{a^2-x^2},$$

$$\therefore \frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\frac{c}{m}}dt,$$

or
$$\sqrt{\frac{c}{m}}t=\sin^{-1}\frac{x}{a}+C_2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where C_2 is the constant of integration. Assuming that we reckon t from the instant at which P starts from A, we see that when $t=0$, $x=a$. Therefore by substitution in (4) we have

$$0=\sin^{-1} 1+C_2, \quad \therefore\ C_2=-\sin^{-1} 1=-\frac{\pi}{2}.$$

Thus as the value of x we obtain

$$x=a \sin\left(\sqrt{\frac{c}{m}}t+\frac{\pi}{2}\right)=a \cos\left(\sqrt{\frac{c}{m}}t\right). \quad . \quad (5)$$

Queries: What are the characteristics of a force producing S.H.M.? In the preceding solution are there other values of C_2 and therefore other values of x?

Equations (3) and (5) give the relation between the velocity and the displacement and between the displacement and the time respectively.

EXERCISE 168. Obtain from equation (5) a relation between the velocity and the time of the above particle.

EXERCISE 169. Test by the theory of dimensions equations (3) and (5), having first deduced the dimensions of c from the equation $F=-cx$.

Undamped Vibrations

According to *Hooke's Law*, that within the elastic limit the strain is proportional to the stress, we may assume that the force exerted by a spring is proportional to the distance that the spring is compressed or elongated. Thus in Fig. 41 let (a) represent a spring when unloaded, (b) the same spring when a weight of W pounds is attached to its end, and (c) the spring when the weight is

displaced by a force of F pounds. Here l is the natural length of the spring, e is the elongation due to the weight W, and $(e+a)$ is the elongation due to the force F plus the weight W.

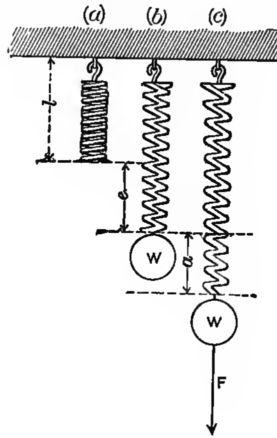


FIG. 41

If T represents the tension of the spring for any elongation, y, then

$$T \propto y,$$

or $$T = cy;$$

but we know that when $T=W$, then $y=e$ by Fig. 41 (b),

$$\therefore\ W = ce \quad \text{or} \quad c = \frac{W}{e},$$

and $T = \frac{W}{e}y$ for any elongation.

The force F can now be found by putting $y=a+e$ and thus obtaining $T=\frac{W}{e}(a+e)$, which is the tension of the spring in (c), and as this tension balances F and W, we have

$$T = \frac{W}{e}(a+e) = F + W,$$

$$\therefore\ F = \frac{Wa}{e}.$$

In the above discussion it must be remembered that the weight W in Fig. 41 is *at rest*.

EXERCISE 170. Assume a spring 3 feet long, fastened at its upper end to a firm support, to be stretched 6 inches by a

weight of 12 pounds. If the weight be now displaced a further distance of 3 inches and released, show that the motion will be simple harmonic and deduce the equations describing the motion.

Exercise 171. Determine the time of a complete vibration of the weight mentioned in Exercise 170.

Exercise 172. Solve Exercise 170, assuming letters instead of numbers for the constants.

Exercise 173. Find the relation between space and time for the motion of a particle of mass *m* released from rest at a distance *a* from a fixed point and subject to a repulsive force proportional to its distance from the fixed point.

Motion Under the Law of Gravitation

A most important law governing variable force is the law of gravitation. The law states that the attraction between any two masses is directly proportional to the product of the masses and *inversely proportional to the square of the distance between their mass-centers.*

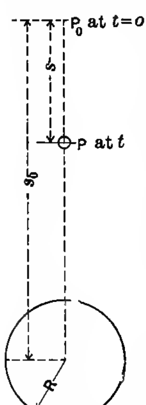


Fig. 42

As an *example,* let us find the velocity which a particle of mass *m*, falling freely from a distance s_0 from the center of the earth, possesses on reaching the earth's surface.

In Fig. 42 let R be the radius of the earth and P_0 the initial position of the particle, while P is its position at any subsequent time t. The acceleration of the particle at P can then be expressed by $\frac{v\,dv}{ds}$, where v is its velocity at P.

At the surface of the earth the force acting upon m would be $W=mg$, its weight. Let F represent the force acting on m at P; then, according to the law of inverse squares,

$$F \propto \frac{1}{(s_0-s)^2} \quad \text{or} \quad F=\frac{c}{(s_0-s)^2};$$

but when $\qquad (s_0-s)=R, \quad \text{then} \quad F=W,$

so that $\qquad W=\frac{c}{R^2} \quad \text{or} \quad c=WR^2,$

and $\qquad F=\frac{WR^2}{(s_0-s)^2}.$

Applying the equation of motion $\Sigma F_y=a_y m$ to the particle we have

$$\frac{WR^2}{(s_0-s)^2}=m\frac{v\,dv}{ds},$$

or

$$\frac{v\,dv}{ds}=\frac{gR^2}{(s_0-s)^2}.$$

Integrating, we have

$$\frac{v^2}{2}=R^2g\int\frac{ds}{(s_0-s)^2}+C=\frac{R^2g}{(s_0-s)}+C.$$

To determine C, note that when $v=0$, $s=0$, thus

$$0=\frac{R^2g}{s_0}+C \quad \text{or} \quad C=-\frac{R^2g}{s_0},$$

or

$$\frac{v^2}{2}=\frac{R^2g}{s_0-s}-\frac{R^2g}{s_0}.$$

At the surface of the earth $s=(s_0-R)$, and we obtain

$$\frac{v^2}{2}=\frac{R^2g}{R}-\frac{R^2g}{s_0}=g\left(R-\frac{R^2}{s_0}\right); \quad \therefore \ v^2=2gR\left(1-\frac{R}{s_0}\right).$$

Exercise 174. From the above equation determine the velocity attained by the particle falling freely from an infinite distance to the surface of the earth. Assume the radius of the earth as 4000 miles.

Exercise 175. From the general value for the velocity

$$v=R\sqrt{\frac{2gs}{s_0(s_0-s)}}$$

derived in the above example obtain the time t required to fall a distance s.

Exercise 176. Determine the time it would take a particle of mass m to reach a center of force attracting it with a force varying inversely as the square of the distance of the particle from the center of force if the particle starts from rest at a distance b from the center.

Exercise 177. A particle moves in a straight line subject to an attraction proportional to $s^{-\frac{3}{2}}$. Show that the velocity acquired in falling from an infinite distance to the distance b is equal to that acquired in falling from rest at b to a distance $\frac{b}{4}$ from the center of attraction.

Section XV

MOTION OF A SYSTEM OF CONNECTED TRANSLATING BODIES

In this section we will consider the accelerations and stresses existing in a system of non-rigidly connected masses whose only motions are translations.

In these problems we will consider the pulleys involved as massless and their pivots as frictionless, so that no force is required to turn them. This is equivalent to saying that the tensions in the strings passing over such imaginary pulleys are equal on both sides of the pulleys.

The method of procedure in problems involving the motion of non-rigidly connected masses is as follows:

1st. Represent the unknown tensions and accelerations by letters.

2d. Find the relation existing between the accelerations involved owing to kinematical reasons. The equations so obtained are called *Kinematic Equations.*

3d. Consider each mass as a "free body" and apply the equation of motion to each. The equations thus found are the *Kinetic Equations.*

4th. See that the number of equations equals the number of unknown quantities. If the equations are too few in number, *Static Equations* must exist at some massless pulley or lever.

5th. Solve the resulting equations.

Example.—Two masses, m and $2m$, are connected by a string passing over a movable pulley, which is attached to a third mass, $3m$, by a string passing over a smooth fixed pulley; find the tensions in the strings and the acceleration of the mass, $3m$.

Solution.—Fig. 43 illustrates the problem. Let the accelerations of the masses, m, $2m$, and $3m$, be a_1, a_2, and a_3, respectively, directed as shown, and let the tensions in the strings be T and T_1.

To obtain the kinematic equations, consider the accel-

erations of the masses m and $2m$ *relative to the pulley* P'. This is equivalent to considering P' at rest, and thus the accelerations of m and $2m$ relative to P' must be equal. As the acceleration of P' is a_3 upward (equal to the acceleration of $3m$), we have

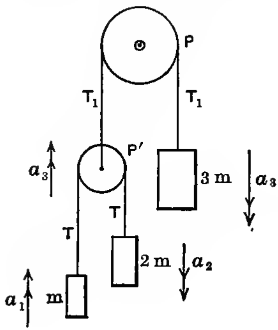


FIG. 43

$$a_2+a_3=a_1-a_3. \quad . \quad (1)$$

Now Fig. 44 shows each mass as a free body. As m, $2m$, and $3m$ are the *masses* in gravitational units of mass, the weights of the respective masses are mg, $2mg$, and $3mg$ pounds. Applying the equations of motion to each mass we have

$$T-mg=ma_1, \quad . \quad . \quad . \quad (2)$$

$$2mg-T=2ma_2, \quad . \quad . \quad (3)$$

and $$3mg-T_1=3ma_3. \quad . \quad . \quad (4)$$

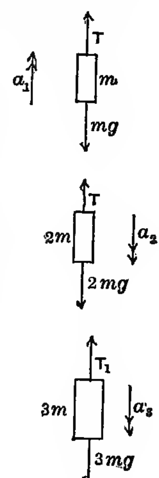


FIG. 44

In equations (1), (2), (3), (4) we have five unknowns, i.e., a_1, a_2, a_3, T, and T_1; we thus need another equation. This equation is found by considering the pulley P'; this being massless, no force is required to accelerate it and thus

$$T_1=2T. \quad . \quad . \quad . \quad . \quad (5)$$

Equations (1) to (5) must now be solved.

Writing (1) in the form

$$2a_3=a_1-a_2$$

and substituting the values of a_1, a_2, and a_3 from equations (2), (3), and (4) in it we have, after eliminating T_1 by means of equation (5),

$$T=\frac{24mg}{17}.$$

By substitution in (2) we find

$$a_3=\frac{g}{17},$$

also
$$T_1=2T=\frac{48mg}{17}.$$

Exercise 178. A weight of 64 pounds is fastened to the end of a string; some distance along the string another mass of 32 pounds is attached. The string then passes over a pulley and to its other end a third mass of 128 pounds is fastened. Find the acceleration of the third mass and the tensions in the various portions of the string.

Exercise 179. A pulley hangs from a spring-balance. A string passing over the pulley has weights of 3 pounds attached to its ends. If a weight of 3 pounds be added to one end of the string, what will be the increase in the pull registered by the spring-balance?

Exercise 180. A body whose weight is 6 pounds rests on a smooth table and is drawn along by a weight of 2 pounds attached to it by a string that passes over a smooth massless pulley at the edge. Find the velocity after 4 seconds.

Exercise 181. Two masses of 100 and 200 pounds respectively are fastened to the ends of a string passing over a weightless pulley. Find the acceleration of the combination and the tension of the string while in motion and when the string is held at the pulley so as to prevent motion.

Exercise 182. Two equal weights are connected by a string 7 feet in length, one of them resting upon a smooth horizontal table 3 feet high at a point 6 feet from the edge where the string passes over a smooth massless pulley to the other weight hanging freely. In what time from rest will the first weight reach the edge of the table?

Exercise 183. Weights of 11 and 5 pounds are suspended from the extremities of a cord which passes over a pulley. What is the velocity of either weight at the end of 5 seconds from rest and the pressure on the support of the pulley?

Exercise 184. Suppose that a stone weighing 100 pounds is placed on a rough horizontal board, the coefficient of statical friction for the stone and board being 0.25. If the board is moved horizontally with an acceleration of 4 ft.-per-sec per sec., will the stone remain on the board?

CHAPTER VI

CONSTRAINED MOTION

Section XVI

REACTION OF THE CONSTRAINING CURVE

A particle is said to be *constrained* in its motion when it is compelled to move along some given fixed curve.

The motion of a particle on an inclined plane already considered in Section XIII is a case of constrained motion. The motion of a ring sliding on a wire (curved or straight), or of a stone attached to one end of a string the other end of which is fixed, are examples of constrained motion.

No new principles are involved in the treatment of constrained motion. The equations of motion may be applied as heretofore, *provided* the body is considered as a "free body". In fact, this must *always* be done. Therefore before applying the equations of motion, show all the forces acting upon the body; these naturally include the reaction of the constraining curve.

Example.—Find the reaction of a smooth constraining curve upon a particle of mass m acted on by forces the sum of whose horizontal and vertical components are X and Y, respectively.

Solution.—In Fig. 45 (a) let MN be the constraining curve and P the particle.

Fig. 45 (*b*) shows the particle as a "free body"; here R represents the reaction of the curve and it is considered positive when directed towards the center of curvature. As the curve is smooth, R is normal to the curve at P, the position of the particle at the time under consideration.

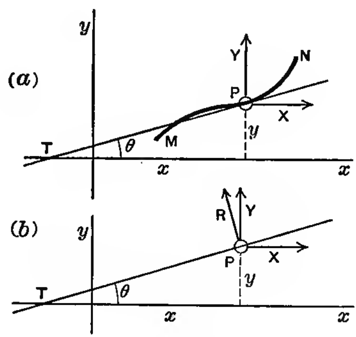


FIG. 45

Let the angle between the tangent PT and the X-axis be θ. Then the equations of motion give

$$X - R \sin\theta = ma_x$$

and

$$Y + R \cos\theta = ma_y.$$

To solve these equations for R, multiply them by $\sin\theta$ and $\cos\theta$, respectively, and subtract the first from the second; then

$$R = m(a_y \cos\theta - a_x \sin\theta) + X \sin\theta - Y \cos\theta.$$

Here $(a_y \cos\theta - a_x \sin\theta)$ is the normal acceleration (a_n) of the particle taken positive when directed towards the center of curvature (see page 40).

$$\therefore\ R = ma_n + X \sin\theta - Y \cos\theta;$$

this is the force exerted by the constraining curve upon the particle.

If no force is acting upon the particle at the instant under consideration, X and Y in the above example would both be zero and the reaction of the curve becomes

$$R = ma_n = m\frac{v^2}{\rho}.$$

Therefore *when no forces act upon the particle* the reaction of the curve is due wholly to the mass of the particle, its velocity, and the radius of curvature of its path. This reaction is necessitated by the inertia of the particle by reason of which it tends to move in a straight line with constant velocity. As the reaction of the curve is always normal to the path it cannot affect the tangential velocity; it only changes the direction of the velocity by producing a normal acceleration.

EXERCISE 185. A particle of mass m moves along the parabola $x^2 = 8y$. When it reaches the vertex it has a velocity of 10 ft. per sec. If no forces act upon the particle at this time, calculate the reaction of the curve upon the particle. What force does the particle exert upon the curve?

EXERCISE 186. If in Exercise 185 the force of gravity be taken into account and the x-axis be horizontal, what is the pressure on the curve?

SECTION XVII

CENTRIPETAL AND CENTRIFUGAL FORCES

The force $R = ma_n$ with which a constraining curve acts upon a particle to keep it in a prescribed path is called the "*centripetal force*" because it always acts towards the center of curvature of the path. In Fig. 46, assume the

particle to move between guides, as shown; then the guides act upon the particle in the several positions, as shown by the arrows. The force exerted by the guides varies with the square of the velocity and the radius of curvature of the path, as indicated by

$$\mathbf{R}=\mathbf{m}\mathbf{a}_n=\mathbf{m}\frac{\mathbf{v}^2}{\rho}.$$

By Newton's third law of motion action and reaction are always equal and opposite in direction. Therefore, if the constraining guide acts upon the particle (Fig. 46)

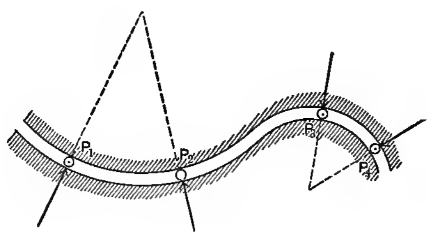


FIG. 46

in a certain direction then the particle reacts upon the constraining guide in the opposite direction with equal force. This force which the particle exerts *upon* the guide is called *centrifugal force.*

Thus the centripetal force acts *upon the particle* while the centrifugal force acts *upon the constraint.*

EXERCISE 187. Find the force necessary to cause a particle weighing 96 pounds to move in a circle whose radius is 3 feet,

(*a*) if the particle has a velocity of 10 feet per second;
(*b*) " " angular velocity of the particle is 6 radians per sec.;
(*c*) " " particle makes 100 revolutions per minute;
(*d*) " " particle moves with a constant velocity and describes the circle in 5 seconds.

Exercise 188. A stone weighing one pound is attached to a string 2 feet long. If the stone rests upon a smooth horizontal plane and the free end of the string is fastened to this plane, find the tension in the string when the stone makes 100 revolutions per second about the fixed end of the string.

Exercise 189. If the string in Ex. 188 will bear a stress of 100 pounds, find the minimum time of one revolution of the stone.

Exercise 190. What is the weight of a body which exerts a force of 10 pounds upon a wire 30 feet long when describing in one hour upon a smooth plane a horizontal circle with the wire as radius?

Exercise 191. A ball weighing 64 pounds revolves in a vertical plane in a circle of 3 feet radius. Find the tension in the cord restraining the ball when the ball is at its highest and lowest position, if the ball makes 20 revolutions per second.

Conical Pendulum

A particle suspended by a weightless thread in such a manner as to allow its moving in a horizontal circle is called a *Conical Pendulum.* Fig. 47 illustrates this; here l is the thread and P the particle. The path of the particle is the horizontal circle whose center is O and whose radius is $OP=r$. While the particle describes this circle the thread generates a cone.

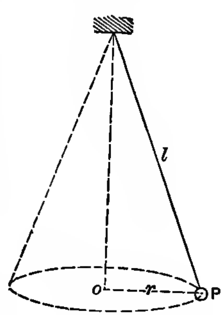


Fig. 47

If the velocity of the particle is v, then the centripetal force necessary to retain it in its circular path must be $m\frac{v^2}{r}$, where m is the mass of the particle, and this force is

directed towards the center of the path, O. Fig. 48 shows the particle as a free body; the forces acting upon it are T, the pull of the thread, and W, the weight of the particle. The resultant of these forces, PC, must be the necessary centripetal force $\left(m\frac{v^2}{r}\right)$ and must act along the radius OP (Fig. 47) towards the center O. This shows that a certain relation exists between T and W, and also between T, W, CP, l, and r. This relation can be obtained by reason of the similarity of the triangles in Figs. 47 and 48.

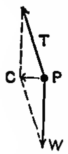


FIG. 48

EXERCISE 192. A conical pendulum whose length is l and the bob of which weighs w pounds revolves n times per second; find the tension in the thread.

EXERCISE 193. Find the semi-angle at the vertex of the cone described by the thread of a conical pendulum whose length is 10 feet, if it revolves 10 times per second. How far from the axis of revolution is the bob? What "deviating" force acts upon the bob?

EXERCISE 194. The revolving balls on a centrifugal governor make 140 revolutions per minute. The distance from the center of each ball to the center of the shaft is 5 inches. The balls are of iron (specific gravity 7) and 2 inches in diameter. Find the centrifugal force of the governor.

Banking of Tracks

A railroad train rounding a curve on a horizontal track exerts a centrifugal force upon the rails and the rails react and supply the necessary centripetal force to the train. On a horizontal track the whole centripetal force is supplied by the lateral pressure of the rails. To avoid this the track is banked.

Fig. 49 shows a section of a banked track. The forces acting upon a train upon a banked track are its weight, W, the normal reaction of the rails, N, supplied through the tread of the wheels, and a force, F, parallel to the banked track acting upon the flanges of the wheels.

The resultant of N, W, and F must be the centripetal force, P, necessary to keep the train moving in its curved path. If a train has a certain velocity, the resultant of

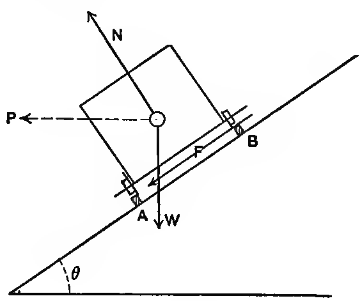


FIG. 49

W and N will be the centripetal force, so that F becomes zero. This is the ideal condition. If the velocity of the train is greater than the velocity for this condition, then F acts as shown in Fig. 49; if less, F acts in the opposite direction.

In banking a track the ideal condition is sought; therefore the inclination is made such that the centripetal force for the mean velocity of the trains is the resultant of the normal pressure of the track, N, and the weight of the train, W.

EXERCISE 195. Find the angle θ (Fig. 49) for the ideal condition of $F=0$ in terms of v, the velocity of the train, and r, the radius of the curve.

Exercise 196. How much should the outer rail be raised on a curve of 600 feet radius if the trains round it at 30 miles per hour, the gauge of the road being 4 feet $8\frac{1}{2}$ inches, if the force exerted by the rails should come upon the tread and not on the flanges of the wheels?

Exercise 197. Find the angle at which the surface of the water in a basin in a railway coach is inclined to the horizontal when the coach rounds a curve of 300 feet radius at a velocity of 20 miles per hour.

Change in Apparent Weight Due to Earth's Rotation

Owing to the rotation of the earth a centripetal force is necessary to keep a body upon its surface. This centripetal force is furnished by a component of the weight of the body. The whole weight of the body is naturally not needed, otherwise bodies upon the earth's surface would have no *apparent weight.* As it is the whole attraction of the earth for a particle, which would be the weight were the earth not rotating, is *not* its apparent weight, but is the apparent weight plus the centripetal force necessitated by the rotation of the earth.

Exercise 198. Assuming the equatorial radius of the earth to be 20,926,000 feet and the time of one revolution upon its axis to be 86,164 seconds, find the deviating force necessary to keep a particle of mass m upon the earth's surface at the equator. If the apparent acceleration due to gravity at the equator is experimentally found to be 32.090 ft.-per-sec. per sec., what would be the acceleration were the earth not rotating?

Exercise 199. Assuming the radius of the earth as R and the time of one rotation as T, find the component of the centripetal force affecting the weight of a particle of mass m, the particle being in latitude θ. Show that owing to the

earth's rotation a certain component of the weight of a particle tends to move it towards the equator.

Section XVIII

CYCLOIDAL PENDULUM

Example.—A particle is constrained to move along any curve MN (Fig. 50) under the action of gravity. Determine its velocity at any time.

Solution.—Assume the particle, whose mass is m, to start at (h, k) with an initial velocity v_0, and determine its velocity at (x, y).

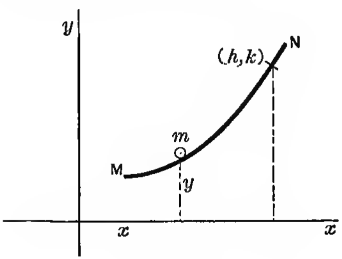


Fig. 50

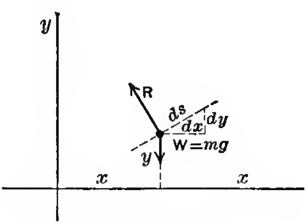


Fig. 51

Fig. 51 shows the particle as a free body; its equations of motion are

$$-R \sin \theta = ma_x,$$

$$R \cos \theta - W = ma_y.$$

Eliminating R from these equations, we have

$$-W \sin \theta = m(a_x \cos \theta + a_y \sin \theta). \quad . \quad . \quad (1)$$

From Fig. 51, note that $\sin \theta = \dfrac{dy}{ds}$ and that $\cos \theta = \dfrac{dx}{ds}$,

where ds is an element of the curve MN. Also $a_x = \frac{d^2x}{dt^2}$ and $a_y = \frac{d^2y}{dt^2}$.

Thus (1) becomes

$$-W\frac{dy}{ds} = m\left\{\frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds}\right\}$$

or
$$-W\,dy = m\left\{\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy\right\}.$$

As
$$\int dx\,d^2x = \frac{dx^2}{2} \quad \text{and} \quad \int dy\,d^2y = \frac{dy^2}{2}$$

we obtain
$$-Wy = \frac{m}{2}\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} + C,$$

and as
$$ds^2 = dx^2 + dy^2 \quad \text{and} \quad \frac{ds}{dt} = v$$

we can put

$$-Wy = \frac{m}{2}\{v^2\} + C.$$

To find C, note that $v = v_0$ when $x = h$ and $y = k$,

$$\therefore\ C = -Wk - \frac{m}{2}v_0^2.$$

The complete integral then becomes

$$Wk - Wy = \frac{m}{2}(v^2 - v_0^2),$$

whence

$$v^2 = 2g(k-y) + v_0^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation shows us that the velocity v of the particle *depends only* upon its *initial velocity*, v_0, and the *vertical height* $(k-y)$ *through which it descends*, and is wholly independent of the nature of the curve followed.

EXERCISE 200. Examine equations (1) and (2) of the above example by the theory of dimensions.

EXERCISE 201. What dimensions must the constant of integration in the above example have?

Example.—Find the pressure exerted by the constraint on a particle of mass m moving within a smooth circular tube bent into a circle whose radius is r and which lies in a vertical plane.

Solution.—Under the above conditions the particle, m, is constrained to move in a circle; let its path be that shown in Fig. 52.

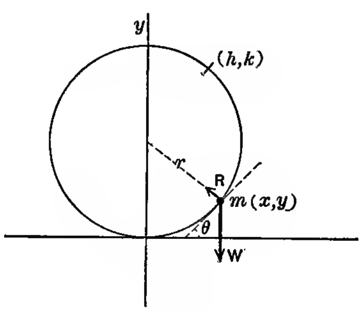


FIG. 52

Here R, the reaction of the tube upon the particle, may be conveniently found by applying the equations of motion along the tangent and normal to the curve. Thus,

$$-W \sin \theta = ma_t,$$

$$R - W \cos \theta = ma_n;$$

$$\therefore\ R = mg\frac{dx}{ds} + m\frac{v^2}{r}.$$

As the equation of the path is $x^2+(r-y)^2=r^2$ or $x^2=2yr-y^2$, we have, on differentiating,

$$x\,dx=r\,dy-y\,dy.$$

$$\therefore\ \frac{dx}{r-y}=\frac{dy}{x}\quad\text{or}\quad\frac{dx^2}{(r-y)^2}=\frac{dy^2}{x^2}.$$

Thus,

$$\frac{dx^2+dy^2}{(r-y)^2+x^2}=\frac{dy^2}{x^2}\quad\text{or}\quad\left(\frac{\sqrt{dx^2+dy^2}}{\sqrt{x^2+(r-y)^2}}=\frac{ds}{r}\right)=\frac{dy}{x}.$$

Therefore $$\frac{dy}{x}=\frac{dx}{r-y}=\frac{ds}{r}.$$

So that $$\frac{dx}{ds}=\frac{r-y}{r}$$ (Prove this by Geometry.)

and $$R=\frac{mv^2}{r}+mg\frac{r-y}{r}.$$

As by the previous example $v^2=2g(k-y)+v_0^2$,

$$R=\frac{mg}{r}\left\{r+2k-3y+\frac{v_0^2}{g}\right\}.$$

Exercise 202. Test the last equation in the above example by the theory of dimensions.

Exercise 203. From $v^2=2g(k-y)+v_0^2$, deduced above, determine (a) the maximum and minimum velocities if the particle makes a complete circuit, (b) the height at which the particle comes to rest if it does not describe the complete circle, (c) the velocity of the particle at any time if it started from rest at the highest point of the tube.

Exercise 204. Deduce the value of R in the above example directly from the equations of motion of the particle with respect to the x- and y-axes.

EXERCISE 205. Find the pressure exerted by the particle on the tube in the above example at the lowest point of its path in terms of the weight of the particle, if (*a*) it starts from rest at the highest point, (*b*) the particle exerts no pressure on the tube at its highest point.

EXERCISE 206. A particle, mass m, slides upon the concave side of a cycloid (Fig. 53). Assuming it to start from rest at (h, k) when $t=0$, determine its velocity and the pressure the particle exerts on the cycloid at any point (x, y).

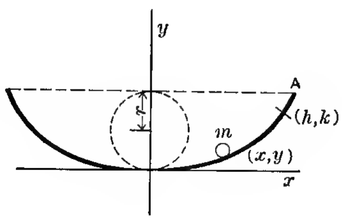


FIG. 53

EXERCISE 207. What pressure will the particle in Exercise 206 exert on the cycloid at its lowest point if it starts at the point A, Fig. 53.

EXERCISE 208. From the velocity determined in Exercise 206 find the time the particle requires to fall from (h, k) to the lowest point of its cycloidal path.

From Exercise 208 we notice that the time required by a particle in falling from any point (h, k) on a cycloid (Fig. 53) to its lowest point is $\pi\sqrt{\frac{r}{g}}$. As this time is independent of h or k, we see that the time of descent does not depend upon the starting-point.

Also, from the value of $v=\sqrt{2g(k-y)}$, as determined in Exercise 206, we see that m will rise to a height k on

the other side of the curve in $\pi\sqrt{\frac{r}{g}}$ seconds from its lowest point. Therefore the time of one oscillation is $2\pi\sqrt{\frac{r}{g}}$ and is independent of the point on the curve at which the motion commences.

Tautochronism is the name given to this property; the cycloid is therefore called the *tautochronous curve* of a heavy particle.

A particle may be made to move in a cycloidal path by the following arrangement. In Fig. 54, suppose the par-

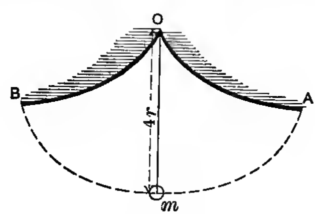


FIG. 54

ticle m to be suspended from O by a weightless inextensible thread of length $4r$, and that the curve AOB represents the outline of two cheeks upon which the thread om may wrap itself as m oscillates. If AOB be the involute of the cycloid AmB (therefore also cycloidal arcs) m will describe the cycloidal path AmB. The particle m will then perform its oscillations in equal times $\left(2\pi\sqrt{\frac{r}{g}}\right)$ no matter what may be its amplitude of vibration. The apparatus just described is known as a *Cycloidal Pendulum.*

SECTION XIX

SIMPLE PENDULUM

Return now to the consideration of a particle moving in a circular path under the action of gravity, Fig. 55.

Example.—What time does a particle require to complete one oscillation if it moves under the action of gravity in a circular path lying in a vertical plane?

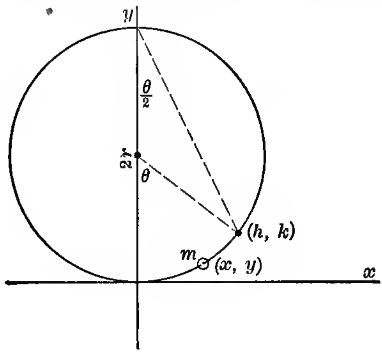


FIG. 55

Solution.—Assume the particle to start from rest at (h, k) and let its mass be m. Its equations of motion will then be

$$-R\frac{dy}{ds}=m\frac{d^2x}{dt^2}$$

and

$$R\frac{dx}{ds}-mg=m\frac{d^2y}{dt^2},$$

where R is the reaction of the constraint.

Eliminating R and integrating between the limits y, k, and 0, v we obtain

$$W(k-y)=\frac{m}{2}(v^2) \quad \text{or} \quad v^2=2g(k-y).$$

And as
$$v=\frac{ds}{dt},$$

we have
$$dt=\frac{ds}{\sqrt{2g}\sqrt{k-y}}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To integrate this we must express the right-hand member in terms of one variable. This is done by means of the equation of the path of the particle, which is

$$x^2+(r-y)^2=r^2.$$

From this we obtain

$$\frac{dx}{r-y}=\frac{dy}{x}=\frac{ds}{r} \qquad \text{(see page 120)},$$

and therefore

$$dt=\frac{r\,dy}{\sqrt{2g}\sqrt{k-y}(x)}=\frac{r\,dy}{\sqrt{2g}\sqrt{(k-y)(2ry-y^2)}}.$$

This expression cannot be integrated by the elementary methods of the Calculus. Its integration depends upon the properties of Elliptic Functions. Under certain conditions an approximate value can be found by expanding the expression into a series and integrating each term separately.

Thus, assume that the particle moves in a large circle and that it never moves far from the x-axis, so that $\frac{y}{2r}$ is a small fraction; then we can expand the integral into ascending powers of $\frac{y}{2r}$ by the Binomial Theorem as follows.

$$t=\frac{1}{2}\sqrt{\frac{r}{g}}\int\frac{dy}{\sqrt{ky-y^2}}\left(1-\frac{y}{2r}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2}\sqrt{\frac{r}{g}}\int\left[1+\frac{1}{2}\cdot\frac{y}{2r}+\frac{1\cdot 3}{2\cdot 4}\left(\frac{y}{2r}\right)^2+\ldots\right]\frac{dy}{\sqrt{ky-y^2}}.$$

Integrating each term separately between the limits 0 and k, we have

$$\frac{T}{2}=\frac{\pi}{2}\sqrt{\frac{r}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{k}{2r}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{k}{2r}\right)^2\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{k}{2r}\right)^3+\ldots\right] \quad . \quad (2)$$

for the time it takes the particle to move from (h, k) to $(0, 0)$, Fig. 55, where T is the time of one oscillation.

And as from Fig. 55 we obtain $\sqrt{\frac{k}{2r}}=\sin\frac{\theta}{2}$, where θ is the semi-angle of the swing, we can write

$$T=\pi\sqrt{\frac{r}{g}}\left[1+\frac{1}{4}\sin^2\frac{\theta}{2}+\frac{9}{64}\sin^4\frac{\theta}{2}+\frac{25}{256}\sin^6\frac{\theta}{2}+\ldots\right]. \quad (3)$$

EXERCISE 209. Complete the indicated steps in the above example.

EXERCISE 210. Test eq. (3) in above example by the theory of dimensions.

The conditions of the above example are fulfilled in the case of a simple or mathematical pendulum. In this contrivance a heavy particle is suspended by a weightless wire which constrains the motion to a circular arc.

As the time of one oscillation [T, eq. (3)] depends

upon the square and higher powers of the sine of the quarter angle through which the particle swings (Fig. 55), we may for small oscillations (3° or less) neglect all but the first term of the series and place

$$T=\pi\sqrt{\frac{r}{g}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where T is the time of one oscillation.

EXERCISE 211. Find the time of one oscillation of a simple pendulum 4 feet long by equation (3), assuming $g=32.172$ and the angle through which it swings as (*a*) 1°, (*b*) = 5°, (*c*) = 30°.

EXERCISE 212. Same as Ex. 211, but by the use of eq. (4).

Compare the values of T obtained by the use of equations (3) and (4).

We thus arrive at the conclusion that *for very small arcs the oscillations of a simple pendulum may be regarded as isochronal.*

Solve the following problems by the use of eq. (4).

EXERCISE 213. If the length of the seconds pendulum at a certain place is 39.04 inches, find the acceleration due to gravity at that place.

EXERCISE 214. If $g=32$, find the number of oscillations a pendulum one foot long will make in one minute.

EXERCISE 215. A clock keeping correct time at a place where $g=32.2$ when moved to another place loses 3 minutes a day; find the value of g at this place.

EXERCISE 216. Assuming r as the radius of the earth, h as the height of a mountain, and n as the number of oscillations which a pendulum beating seconds at the earth's surface makes at the top of the mountain in 24 hours, show that

$$h=\left\{\frac{24\times 60\times 60}{n}-1\right\}r.$$

Hint.—Show by the law of gravitation that $\frac{g}{g_1}=\left(\frac{r+h}{r}\right)^2$, where g and g_1 are the accelerations due to gravity at the earth's surface and at the top of the mountain respectively. Then by means of eq. (4) arrive at the required result.

EXERCISE 217. If the variation of the acceleration of gravity follows the law $\frac{g}{g'}=\frac{r}{r-h}$, where g and g' are the accelerations at the surface of the earth and at the bottom of a mine, respectively, show that h, the depth of the mine, is

$$h=\left\{1-\left(\frac{n}{24\times 60\times 60}\right)^2\right\}r,$$

where r is the earth's radius and n is the number of oscillations in 24 hours of a pendulum beating seconds at the mouth of a mine when taken to the bottom.

CHAPTER VII

KINETICS OF A RIGID BODY

Section XX

TRANSLATION OF A RIGID BODY

As already explained in Section VII the translation of a body is determined by the motion of any one of its points. In Section XII it was shown that the acceleration of the mass-center of a body is always such as would be experienced by a particle occupying its position and whose mass equals the mass of the body and to which are applied forces equal and parallel to the forces acting upon the body.

Thus no matter how complicated the motion of a body may be the translation of its mass-center can always be determined by the principles already discussed for the motion of a particle.

If any body, for example a chair, be thrown from a window, the motion of any point will in general be most confusing to follow. The chair turns over and over, and any given point will describe a complicated path. The explanation of the motion of this point will at once be simplified if we note that the mass-center of the chair moves in a parabolic path (similar to that described by a particle under like conditions) and that the point under

observation rotates about a moving axis passing through the mass-center of the chair.

The equations of motion for the mass-center of a rigid body, as deduced on page 84, are

$$\Sigma \mathbf{F}_x = \mathbf{a}_x \Sigma \mathbf{m},$$

$$\Sigma \mathbf{F}_y = \mathbf{a}_y \Sigma \mathbf{m}.$$

EXERCISE 218. A locomotive exerts a draw-bar pull of 900 pounds upon a train weighing 50 tons. Assuming the frictional resistance to be 15 pounds per ton and the air resistance 2 pounds per ton, what will be the acceleration of the train and how far will it travel from rest in 2 minutes?

EXERCISE 219. A freight train of 100 tons weight moves at the rate of 30 miles per hour when the steam is shut off and the brakes are applied to the locomotive. Assuming that the weight of the locomotive is 20 tons and the only friction that of the locomotive, what is the frictional force per ton if the train stops after moving 2 miles?

SECTION XXI

ROTATION OF A RIGID BODY

The law governing the rotation of a rigid body about a fixed axis will now be considered.

Let Fig. 56 represent a body composed of particles $m_1, m_2, m_3, \ldots,$ located at distances $r_1, r_2, r_3, \ldots$ from the axis of rotation, O.

Then if α be the angular acceleration of the body about the axis O, the tangential acceleration (a_t) of each particle will be $a_1 = \alpha r_1$, $a_2 = \alpha r_2$, $a_3 = \alpha r_3, \ldots$ (see page 58).

If each particle is now considered separately we obtain the forces (acting upon the particles) necessary to pro-

duce their respective tangential accelerations by means of the formula $F=ma$. They are

$$F_1=m_1a_1;\quad F_2=m_2a_2;\quad F_3=m_3a_3;\quad \text{etc.}$$

As the forces F_1, F_2, F_3, etc., act at different distances from the axis of rotation, O, their total effect cannot be obtained by adding them. We must obtain their moments about O, and then, by adding their moments, we

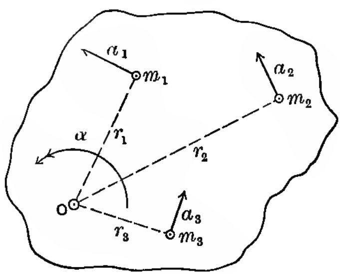


FIG. 56

obtain their total tendency to produce rotation. Thus $F_1r_1=r_1m_1a_1$, $F_2r_2=r_2m_2a_2$, etc., are their moments, and the sum of their moments is

$$\begin{aligned} F_1r_1+F_2r_2+\ldots &= r_1m_1a_1+r_2m_2a_2+\ldots \\ &= r_1m_1\alpha r_1+r_2m_2\alpha r_2+\ldots \\ &= \alpha r_1^2m_1+\alpha r_2^2m_2+\ldots \end{aligned}$$

If we now place $F_1r_1+F_2r_2+F_3r_3+\ldots$, the total moment producing the rotation of the body (which is sometimes called the *torque*), equal to $\Sigma Fr=\Sigma Mo$, we have

$$\begin{aligned} \Sigma Mo &= \alpha r_1^2m_1+\alpha r_2^2m_2+\alpha r_3^2m_3+\ldots \\ &= \alpha(r_1^2m_1+r_2^2m_2+r_3^2m_3+\ldots)=\alpha\Sigma r^2m. \end{aligned}$$

Thus we see that the idea of rotation introduces us to three physical quantities ΣMo, α, and Σr^2m. We are already familiar with the first two, the sum of the *moments of the forces* acting on the body and the *angular acceleration* produced.

The quantity $\Sigma \mathbf{r^2m}$ is, however, a new concept. This quantity was first introduced into Mechanics by Euler, and he named it "*Moment of Inertia*". This name it has retained. A better name might be moment of mass.

The moment of inertia of a body about any axis *can be defined as the sum of the products formed by multiplying each element of mass by the square of its distance from the axis.*

The moment of inertia of a body is usually designated by the letter I.

We may now write the *equation of motion for rotation* as follows:

$$\Sigma \textbf{ Moments} = \mathbf{I}\alpha$$

This equation should be compared with the formula $F = ma$ and the resemblance noted.

For translation and rotation we have respectively

$$\left.\begin{array}{l}\text{force}\\ \text{mass}\\ \text{acceleration}\end{array}\right\}\text{and}\left\{\begin{array}{l}\text{moment of force (torque),}\\ \text{moment of inertia (moment of mass),}\\ \text{angular acceleration.}\end{array}\right.$$

EXERCISE 220. What are the dimensions of (*a*) Torque? (*b*) Moment of Inertia? Can force and torque be measured in the same units? Can mass and moment of inertia be measured in the same units?

EXERCISE 221. Define the gravitational unit of (*a*) torque; (*b*) moment of inertia.

EXERCISE 222. A pulley having a radius of 3 feet, and capa-

ble of rotating about a fixed axis, has a moment of inertia of 20 units; it is set in rotation by a force of 10 pounds applied to a rope coiled about its circumference. Neglecting all friction,

(*a*) what will be its angular acceleration?

(*b*) what will be its angular velocity after 4 seconds if it starts from rest?

(*c*) how many revolutions will it make in 2 minutes starting from rest?

EXERCISE 223. If in the preceding exercise the force ceases to act after 6 seconds, what will be the velocity of rotation in revolutions per minute? How long will the body retain this velocity?

EXERCISE 224. A belt passes over a pulley. The tension on one side of the pulley is 100 pounds; on the other 90 pounds. If the moment of inertia of the pulley is 40 units and its diameter is 26 inches, how many revolutions per minute will the pulley be making 5 seconds after starting from rest?

EXERCISE 225. Assume that two masses of 320 pounds each, one concentrated at each end of a weightless rod 6 feet long, be caused to rotate about an axis perpendicular to the rod at its center by a force of 100 pounds acting perpendicularly to the rod in the plane of motion and at a distance of 2 feet from the axis. Find the angular acceleration.

What would be the angular acceleration (*a*) if the masses were 2 feet apart, (*b*) if the rod were 10 feet long?

EXERCISE 226. A fly-wheel, weighing W pounds and whose moment of inertia is I, makes n revolutions per minute when the driving force F, applied to its circumference, ceases to act. If μ is the coefficient of friction and the radius of the axle is r feet, find the time in which the wheel will come to rest.

EXERCISE 227. How many revolutions will the fly-wheel of Exercise 226 make before coming to rest?

SECTION XXII

ON MOMENT OF INERTIA

In order to apply the equation of motion for rotation the moment of inertia of the rotating body must be known.

Moments of inertia can most readily be found by means of the Calculus. The general method of procedure is to select a differential element of the body and write the expression for the product of its mass by the square of its distance from the axis of rotation. This expression should then be integrated so as to include all the elements of the body. This is simply a direct application of the definition given on page 131.

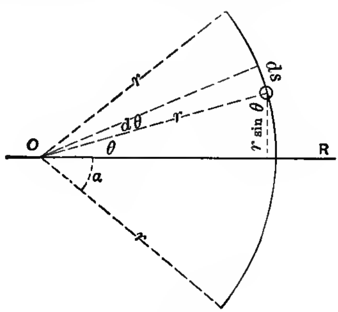


FIG. 57

Example.—Find the moment of inertia of a uniform thin wire bent in the form of a circular arc and revolving about a diameter, bisecting its chord.

Solution.—Assume the cross-section of the wire, which we assume infinitesimal, to be a, and let the radius of and the semi-angle subtended by the arc be r and α respectively (Fig. 57).

Consider any element of the wire, as ds, whose principal point is at a distance of $r \sin \theta$ from the axis of rotation OR.

As (mass) = (volume)(density) the mass of this element is $ads\ \delta$, where δ is the density of the wire. Therefore the moment of inertia of the element is

$$r^2 \sin^2\theta\ ads\ \delta.$$

To integrate this expression it must be expressed in terms of a single variable, say θ; as $ds = rd\theta$, we have

$$r^2 \sin^2\theta\ ard\theta\ \ \delta.$$

If I represents the moment of inertia of the whole wire

$$I = ar^3\delta \int_{-\alpha}^{+\alpha} \sin^2\ \theta\ d\theta = \frac{ar^3\delta}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\alpha}^{+\alpha}$$

$$= \frac{ar^3\delta}{2}\{2\alpha - \sin 2\alpha\},$$

which is the moment of inertia sought.

Exercise 228. A straight wire revolves about an axis through its end and perpendicular to its length. Assuming l, a, and δ as the length, section, and density of the wire respectively, find I.

Exercise 229. Find I for the wire shown in Fig. 58 about O as an axis.

Exercise 230. Find I for a wire bent into a complete circle of radius r and rotating about an axis through its center and perpendicular to its plane. Show that I is equivalent to the moment of inertia of the whole mass of the wire concentrated at any point in the circle.

Example.—Determine I for a lamina in the form of a quadrant of a circular lamina rotating about one of its straight sides.

Solution.—Assume t as the thickness of the lamina, which we will consider infinitesimal, and δ as its density. Let a be the radius of the circle.

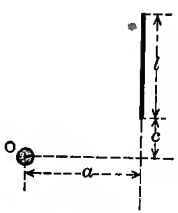


FIG. 58

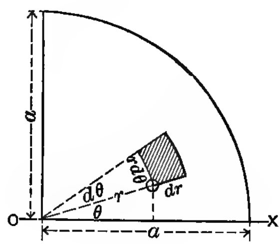


FIG. 59

As in Fig. 59, assume any element of the lamina. Its mass is $(rd\theta\, dr)(t\delta)$ and its moment of inertia about OX is

$$(rd\theta\, dr)(t\delta)r^2 \sin^2 \theta.$$

$$\therefore\ I = t\delta \int_0^a r^3 dr \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = t\delta \int_0^a \frac{r^3 dr}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi t\delta}{4}\int_0^a r^3 dr = \frac{\pi t\delta a^4}{16}.$$

EXERCISE 231. Solve the preceding example using rectangular coordinates.

EXERCISE 232. Find the I of a thin rectangular lamina $h \times b$ about an axis through the center of gravity,

(*a*) parallel to b,

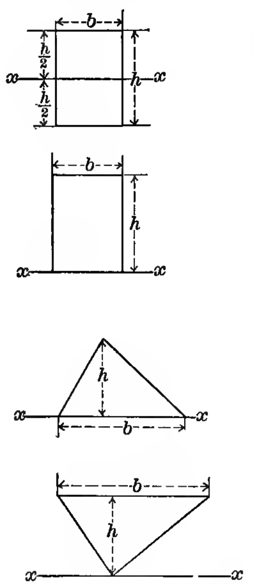
(*b*) parallel to h.

EXERCISE 233. Same as Ex. 232, but (*a*) about the side h as an axis, (*b*) about the side b as an axis.

EXERCISE 234. Find the I of a triangular lamina of base b and altitude h about the base.

Hint.—Assume the segments into which the altitude divides the base to be m and n.

The moments of inertia for the laminæ shown in Fig. 60 are found by integration to be those given in the figure.



$$I = \frac{t\delta}{12} h^3 b$$

$$I = \frac{t\delta}{3} h^3 b$$

$$I = \frac{t\delta}{12} h^3 b$$

$$I = \frac{t\delta}{4} h^3 b$$

FIG. 60

By means of these the I of rectilinear figures may be found by dividing them into rectangles and triangles and taking the sum of the moments of inertia of the separate parts.

EXERCISE 235. Find the I of the laminæ shown in Fig. 61 by dividing the plates into rectangles and triangles and applying the results shown in Fig. 60.

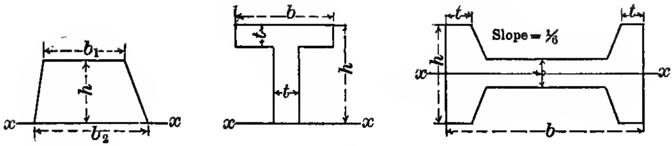


FIG. 61

Moments of Inertia about Parallel Axes

Theorem.—The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the mass-center of the body plus the product of the mass of the body and the square of the distance between the axes.

Proof.—In Fig. 62, assume the mass-center of the body at G and the axis at O perpendicular to the plane of the paper.

We wish to show that $I_0 = I_g + d^2\Sigma m$, where I_0 and I_g are the moments of inertia about the axis through O and G, respectively, and Σm is the mass of the body.

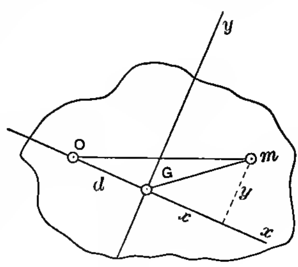


FIG. 62

Pass the X-axis through O and G and the Y-axis through G, and consider any element m of the body. Then

$$I_0 = \Sigma\{(x+d)^2 + y^2\}m = \Sigma\{x^2+y^2\}m + \Sigma d^2 m + 2d\Sigma mx$$

and
$$I_g = \Sigma\{x^2+y^2\}m.$$

If $\bar{x}$ is the abscissa of the mass-center, then $\bar{x} = \dfrac{\Sigma mx}{\Sigma m} = 0$ by the choice of the position of the axis,

$$\therefore\ \Sigma mx = 0.$$

It follows that $$\mathbf{I}_0=\mathbf{I}_g+\mathbf{d}^2\Sigma\mathbf{m},$$

which proves the theorem. Prove this theorem when the axes lie in the plane of the base.

EXERCISE 236. From Exercises 228 and 229, by means of the above theorem, find the I of the wire about a parallel axis through its mass-center.

EXERCISE 237. From Ex. 234 find the I about the axis parallel to the base through the mass-center.

EXERCISE 238. From the first result given in Fig. 60 find the I of the rectangle about an axis parallel to XX, a units above XX.

EXERCISE 239. From the third result, Fig. 60, obtain the fourth by the use of the above theorem.

Polar Moments

Theorem.—The moment of inertia of a lamina about an axis, through any point O, and perpendicular to its plane (*polar moment*) is equal to the sum of the moments of inertia about any two rectangular axes through O and in the plane of the lamina (*rectangular moments*), or

$$\mathbf{I}_p=\mathbf{I}_x+\mathbf{I}_y.$$

EXERCISE 240. Prove the above theorem.

EXERCISE 241. Find by integration the polar moment of a circular lamina about an axis through its center.

EXERCISE 242. Find two rectangular moments for the lamina of Ex. 241 and from them deduce the result of Ex. 241.

EXERCISE 243. Find the polar moment of inertia about an axis through the center of gravity of a rectangular lamina $h\times b$.

Principal Axes for a Point in a Lamina

The principal axes for any point in a lamina are the axes passing through the point and lying in the plane of the lamina, for which the moment of inertia of the lamina has a maximum and minimum value respectively.

In Fig. 62 (*a*) assume a set of rectangular axes (xy) through the point O. We will now show that these axes

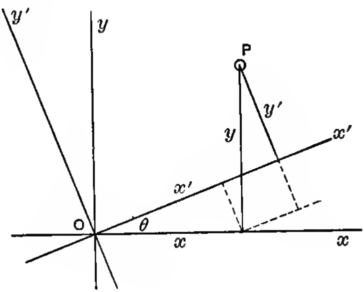


FIG. 62 (*a*)

are the principal axes if $\Sigma mxy = 0$, where m is the mass of any particle P.

Assume any other set of axes ($x'y'$) inclined to the original set at an angle θ. Then from the figure the co-ordinates of the particle P with reference to the new axes are

$$\left.\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ \text{and} \qquad y' &= y \cos\theta - x \sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus the moment of inertia about the x'-axis is

$$\begin{aligned} I_{x'} &= \Sigma my'^2 \\ &= \Sigma my^2 \cos^2\theta - 2\Sigma mxy \sin\theta\cos\theta + \Sigma mx^2 \sin^2\theta, \end{aligned}$$

and as by hypothesis $\Sigma mxy = 0$,

$$\mathbf{I}_{x'} = \mathbf{I}_x \cos^2\theta + \mathbf{I}_y \sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where I_x and I_y are to be proved the principal moments of inertia. From (2) we see that if $I_x > I_y$, then, in general, $I_{x'}$ will be less than I_x and greater than I_y, so that I_x is a maximum and I_y a minimum value of $I_{x'}$ for varying values of θ. Prove this by the calculus.

What happens if $I_x = I_y$?

To find the principal axes of a lamina, assume any set of axes and find Σmx^2, Σmy^2, and Σmxy. Then by means of (1) find $\Sigma mx'y'$ for another set of axes of unknown inclination, θ, to the original set. Thus

$$\Sigma mx'y' = \Sigma mxy \cos 2\theta - \tfrac{1}{2}\Sigma m(x^2 - y^2) \sin 2\theta.$$

Put this equal to zero and find

$$\tan 2\theta = \frac{2\Sigma \mathrm{mxy}}{\Sigma \mathrm{mx}^2 - \Sigma \mathrm{my}^2}, \quad . \quad . \quad . \quad . \quad (3)$$

which gives the value of θ.

Therefore the principal axes $(x'y')$ are inclined to the assumed axes (xy) at the angle θ found from (3).

Radius of Gyration

We have seen how the moment of inertia of a system of particles about an axis is equal to Σmr^2.

Suppose we could concentrate the mass of all the particles at one point and that we desire that the moment of inertia of this concentrated mass, Σm, about the same axis be still Σmr^2, then the concentrated mass must be

placed at a certain distance from the axis. This distance, k, is called the *radius of gyration.* In accordance with the above conditions we may put

$$k^2 \Sigma m = \Sigma m r^2 = I,$$

$$\therefore \ \mathbf{k}^2 = \frac{\mathbf{I}}{\Sigma \mathbf{m}},$$

or, *the square of the radius of gyration is equal to the moment of inertia divided by the mass of the body.*

EXERCISE 244. Find the radius of gyration for the wires described in Exercises 228 and 229.

EXERCISE 245. What is the value of k^2 for the laminæ described in Exercises 232 and 234?

The radius of gyration about an axis through the mass-center of a body is called the *principal radius of gyration; this we will denote by* $\bar{k}$.

EXERCISE 246. Show that $\mathbf{k}^2 = \bar{\mathbf{k}}^2 + \mathbf{d}^2$ if the distance d separates the parallel axes about which k and $\bar{k}$ are taken.

Reduction of Mass.—If I is the moment of inertia of a body whose mass is m, then as far as the angular acceleration produced by certain moments is concerned we may consider the whole mass m concentrated at a distance equal to the radius of gyration, k, from the axis. Sometimes it is more convenient to replace the mass m by an *equivalent concentrated mass at some other radius.* This is called the reduction of the mass. As the moment of inertia must not change, we have

$$\mathbf{I} = \mathbf{m}\mathbf{k}^2 = \mathbf{m}_1 \mathbf{k}_1{}^2,$$

where m_1 is the concentrated mass to be placed at a distance k_1 from the axis of rotation.

EXERCISE 247. Find the reduced mass for the lamina described in Ex. 232 at a radius r from the axis of rotation.

EXERCISE 248. A circular plate weighing 32 pounds per sq. ft., and whose radius is 2 feet, revolves about a diameter as an axis. Find the mass which when concentrated at a distance of 2 feet from the axis has the same moment of inertia as the plate. What concentrated mass at 1 foot radius would be required to replace the plate? What concentrated mass at a 10-foot radius?

EXERCISE 249. A mass of 100 pounds is concentrated at a distance of 3 feet from an axis of rotation. What will be the weight of the equivalent mass placed at a radius of (*a*) 0.5 foot? (*b*) 1 foot? (*c*) 10 feet? What is the moment of inertia of the mass?

Example.—Find the moment of inertia of a cylinder revolving about its axis.

Solution.—Assume the radius, altitude, and density of the cylinder as a, h, and δ, respectively.

Assume an element using cylindrical coordinates (Fig. 63); then the moment of inertia of an element is

$$(\delta r\,d\theta\,dr\,dy)r^2,$$

and

$$I=\delta\int_0^h dy\int_0^a r^3dr\int_0^{2\pi}d\theta,$$

$$I=\frac{\delta 2\pi a^4h}{4}=\frac{\pi a^4h\delta}{2}.$$

EXERCISE 250. Show that the radius of gyration of the cylinder of the above example is $k=\frac{a}{2}\sqrt{2}$.

EXERCISE 251. Show that the radius of gyration of a sphere of radius a about a diameter is $a\sqrt{\frac{2}{5}}$.

EXERCISE 252. Show that $k^2 = \frac{3d^2}{40}$ for a right cone of altitude h and diameter of base d when revolving about its axis.

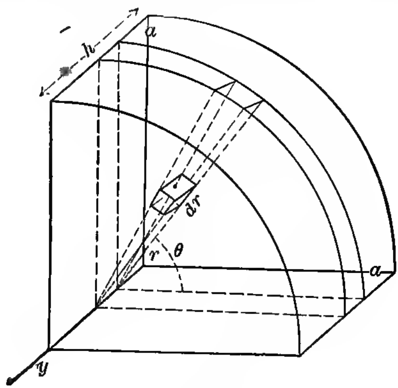


FIG. 63

EXERCISE 253. Show that the radius of gyration of a hollow sphere revolving about a diameter is

$$\sqrt{\frac{2\{(\text{outer radius})^5 - (\text{inner radius})^5\}}{5\{(\text{outer radius})^3 - (\text{inner radius})^3\}}}.$$

APPLICATIONS OF THE EQUATION OF MOTION FOR ROTATION

Section XXIII

ROTATION DUE TO CONSTANT FORCES

We shall now apply the equation of motion for rotation to bodies revolving about a fixed axis.

Example.—A body is made to rotate about a horizontal axis passing through its mass-center by a mass, whose weight is W pounds, fastened to a cord wrapped about a cylindrical portion of the body of radius r. Assuming I as the M. of I. of the body about its mass-center and neglecting all friction, find the angular acceleration of the body and the tension in the cord. Also find the angle through which it will turn in t seconds if it starts from rest.

Solution.—Fig. 64 (a) shows the rotating body and the weight W; the center of rotation is at C, the mass-center. Each body should now be made a "free body", as in Figs. 64 (b) and (c). Here W_1 represents the weight of the rotating body, R the reaction of the bearing, and T the tension in the cord.

Let α represent the angular acceleration of the body about its mass-center taken positive in the direction shown by the arrow, and let a represent the acceleration of W taken positive when downward.

From Fig. 64 (b) we have, by the equation of motion for rotation applied to rotation about C,

$$Tr = I\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and from Fig. 64 (c) by the equation of motion for translation,

$$W-T=\frac{W}{g}a. \qquad (2)$$

These equations contain three unknown quantities, α, a, and T; thus we need another equation. This is obtained

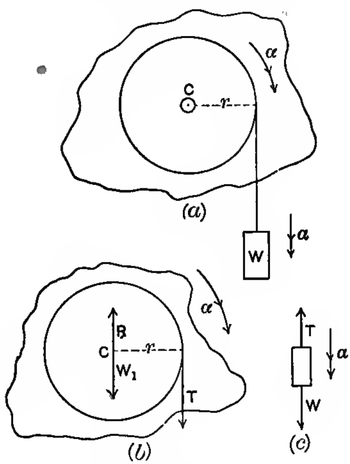


FIG. 64

from Fig. 64 (a) by the principles of kinematics (p. 58). It is

$$a=\alpha r. \qquad (3)$$

Solving equations (1), (2), and (3), we obtain

$$\alpha=\frac{Wr}{I+\frac{W}{g}r^2} \quad \text{and} \quad a=\frac{Wr^2}{I+\frac{W}{g}r^2},$$

and
$$T=\frac{I\alpha}{r}=\frac{IW}{I+\frac{W}{g}r^2}.$$

To find the angular displacement in a given time we may express α in its differential form $\frac{d^2\theta}{dt^2}$, integrate twice and determine the constants by means of the initial conditions of the motion; or, if we remember the kinematical formulæ for angular motion with *constant acceleration*, namely, $\omega = \alpha t + \omega_0$, $\theta = \frac{1}{2}\alpha t^2 + \omega_0 t$, and $\omega^2 = 2\alpha\theta + \omega_0^2$, we may use the second formula, for α in this example equals $\dfrac{Wr}{I + \dfrac{W}{g}r^2}$, which is a constant.

EXERCISE 254. Test the value of α and T of the above example by the theory of dimensions.

EXERCISE 255. If the mass of the rotating body and of the translating body in the above example are each equal to m, find the values of α and T in terms of the radius of gyration, k, of the rotating body instead of I.

EXERCISE 256. Find by integration the value of θ for Ex. 255.

EXERCISE 257. A cylinder (radius 5 feet, length 2 feet) composed of iron (specific gravity 7) is made to rotate about its axis by a force of 100 pounds applied to a cord wrapped around its cylindrical surface. Find the angular acceleration produced and the angular velocity generated by the force in one minute.

(See Ex. 250 for k and thus find I.)

EXERCISE 258. Find the angular velocity generated by the force in Exercise 257, if applied to a cylinder of equal mass but with a radius of 20 feet.

EXERCISE 259. Same as Ex. 257, but instead of a force of 100 pounds let a mass whose weight is 100 pounds be attached to the cord.

EXERCISE 260. Find the tension in the cord in Ex. 259.

EXERCISE 261. A sphere of mass M is caused to rotate about a diameter as an axis by a mass of m attached to a cord wound about the great circle perpendicular to the axis. If the radius of the sphere is r, find the angular acceleration produced. (See Ex. 251 for k.)

EXERCISE 262. A wheel and axle of mass m and radius of gyration k has masses of m_1 and m_2 suspended from it by cords wrapped around its large and small circumferences of radii R and r, respectively. Neglecting all friction, find the angular acceleration of the wheel and axle and the linear acceleration of the mass m_1. How far will the mass m_1 descend in t seconds?

Example.—A wheel of radius r and weight W has a M. of I., I, about its center. Find the angular velocity of

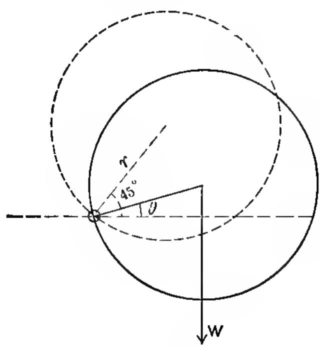


FIG. 65

the wheel at any time if it revolves about an axis tangent to its circumference and perpendicular to its plane under the action of gravity and starts with the radius to its center 45° above the horizontal through the axis.

Solution.—Fig. 65 illustrates the problem. The dotted circle shows the starting position and the full circle the

position of the wheel t seconds after starting. The equation of motion $\Sigma M = I\alpha$ gives

$$-Wr\cos\theta = \left(I + \frac{W}{g}r^2\right)\alpha,$$

where $I + \frac{Wr^2}{g}$ is the M. of I. of the wheel about the axis O.

$$\therefore\ \alpha = -\frac{Wr}{I + \frac{W}{g}r^2}\cos\theta,$$

but
$$\alpha = \frac{\omega\, d\omega}{d\theta};$$

$$\therefore\ \omega\, d\omega = -\frac{Wr}{I + \frac{W}{g}r^2}\cos\theta\, d\theta,$$

or
$$\frac{\omega^2}{2} = -\frac{Wr}{I + \frac{W}{g}r^2}\sin\theta + C.$$

The starting conditions show that when $\theta = \frac{\pi}{4}$, $\omega = 0$,

$$\therefore\ C = \frac{Wr}{I + \frac{W}{g}r^2}\left(\frac{1}{\sqrt{2}}\right);$$

so that
$$\omega^2 = \frac{Wr}{I + \frac{W}{g}r^2}\{\sqrt{2} - 2\sin\theta\}.$$

EXERCISE 263. Find the angular velocity of the wheel in the above example when $\theta = -45°$, $-90°$, $-180°$, and $-225°$ if it starts with its center on a level with the axis.

EXERCISE 264. Find the general expression for the angular acceleration of the wheel in the above example if a cord wrapped around its circumference carries a weight, W, at its free end.

COMPOUND OR PHYSICAL PENDULUM

An important application of the principle of rotation is found in the motion of a heavy body rotating about a horizontal axis under the action of gravity and the reaction of the axis. A body oscillating about a horizontal axis without making a complete revolution is known as

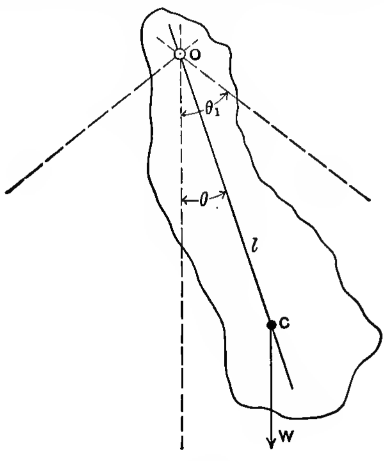


FIG. 66

a compound pendulum, the word compound being used in distinction to simple, as applied to the ideal pendulum already studied in Section XIX.

Let Fig. 66 represent a compound pendulum. Here O is the "center of suspension" and C the centroid of the pendulum.

Let l denote the distance OC;

k, the radius of gyration of the body with respect to O as an axis;

θ, the angle the line OC makes with the vertical at any time t;

θ_1, the maximum angular displacement;

W, the weight; and

m, the mass of the pendulum.

Assuming counter-clockwise as the positive direction the equation for rotation, $\Sigma M = I\alpha$, gives

$$-Wl \sin \theta = k^2 m\alpha,$$

but as $$\alpha = \frac{d^2\theta}{dt^2} \quad \text{and} \quad W = mg,$$

$$\frac{d^2\theta}{dt^2} = -\frac{gl}{k^2} \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This expression if integrated yields the values of ω and θ for any time t.

Multiplying both members of the differential equation by $d\theta$ we have

$$d\theta \frac{d^2\theta}{dt^2} = -\frac{gl}{k^2} \sin \theta \, d\theta.$$

Integrating we have

$$\frac{1}{2} \frac{d\theta^2}{dt^2} = \frac{gl}{k^2} \cos \theta + C.$$

Note that $\frac{d\theta}{dt} = \omega =$ the angular velocity, and as $\omega = 0$ when $\theta = \theta_1$, we obtain

$$C = -\frac{gl}{k^2} \cos \theta_1,$$

$$\therefore \omega = \frac{d\theta}{dt} = \pm \frac{\sqrt{2gl}}{k} \sqrt{\cos \theta - \cos \theta_1}. \quad . \quad . \quad (2)$$

From this equation the oscillatory nature of the motion is evident, for when $\theta=\theta_1$, $-\theta_1$, $2\pi+\theta_1$, $2\pi-\theta_1$, etc., ω becomes zero and the body is momentarily at rest. The angle of oscillation is thus $2\theta_1$.

To find the time of oscillation it is necessary to integrate eq. (2). Separating the variables we have

$$dt=\frac{k}{\sqrt{2gl}}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_1}}. \quad . \quad . \quad . \quad . \quad (3)$$

The resulting function of θ cannot be integrated by elementary methods; it is called an elliptic integral. Approximate results may, however, be obtained by expanding $\cos\theta$ and $\cos\theta_1$ into series if we limit the displacement of the body to small angles so that θ and θ_1 are both small.

Under this assumption put $\cos\theta=1-\frac{\theta^2}{2}$ and $\cos\theta_1=1-\frac{\theta_1^2}{2}$.

Then (3) becomes

$$dt=\frac{k}{\sqrt{gl}}\frac{d\theta}{\sqrt{\theta_1^2-\theta^2}},$$

$$\therefore \; t=\pm\frac{k}{\sqrt{gl}}\sin^{-1}\frac{\theta}{\theta_1}+C_1.$$

If now we measure the time from the instant at which the centroid of the pendulum passes the vertical line through the axis, then $t=0$ when $\theta=0$. $\therefore \; C_1=0$.

To determine which of the signs ($\pm$) should be used, assume the pendulum to be moving in the positive direc-

tion when $t=0$. Then θ increases positively and t increases positively.

$$\therefore\ t=+\frac{k}{\sqrt{gl}}\sin^{-1}\frac{\theta}{\theta_1}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From eq. (4) the period of oscillation can be found. Let t_1 be the time at which the pendulum reaches its maximum negative displacement; then when $t=t_1$, $\theta=-\theta_1$.

$$\therefore\ t_1=\frac{k}{\sqrt{gl}}\sin^{-1}(-1)=\frac{k}{\sqrt{gl}}\left(\frac{3\pi}{2}\right)=\frac{k}{\sqrt{gl}}\left(\frac{7\pi}{2}\right)=\dots$$

If t_2 is the time at which the pendulum reaches its maximum positive displacement, then $t=t_2$ when $\theta=+\theta_1$,

$$\therefore\ t_2=\frac{k}{\sqrt{gl}}\sin^{-1}(1)=\frac{k}{\sqrt{gl}}\left(\frac{\pi}{2}\right)=\frac{k}{\sqrt{gl}}\left(\frac{5\pi}{2}\right)=\dots$$

From these results note that the pendulum *first* reaches its maximum positive displacement $\frac{k}{\sqrt{gl}}\left(\frac{\pi}{2}\right)$ seconds after passing the vertical and reaches its maximum negative displacement for the first time after $\frac{k}{\sqrt{gl}}\left(\frac{3\pi}{2}\right)$ seconds.

Therefore the *time of one oscillation* is $\frac{k}{\sqrt{gl}}(\pi)$, or

$$\mathbf{T}=\pi\sqrt{\frac{\mathbf{k}^2}{\mathbf{gl}}}.$$

EXERCISE 265. Deduce the time of oscillation for a compound pendulum from equation (1) of the above solution by noting that for small angles the sine and the angle are practically equal; thus put $\sin\theta=\theta$ and integrate.

From Section XIX it will be remembered that the time of one oscillation of a *simple* pendulum is $\pi\sqrt{\frac{r}{g}}$, where r denotes the length of the pendulum.

EXERCISE 266. Find the length of a simple pendulum having the same period as a compound pendulum whose radius of gyration about the axis of suspension is k and whose centroid is at a distance l from the axis.

From Exercise 266 we find the equivalent length of the simple pendulum to be $r=\frac{k^2}{l}$. This leads to the conclusion illustrated in Fig. 67. Here we assume the whole mass of the compound pendulum to be concentrated at M on a line passing through the axis and the centroid, C, and at a distance r $\left(=\frac{k^2}{l}\right)$ from the axis.

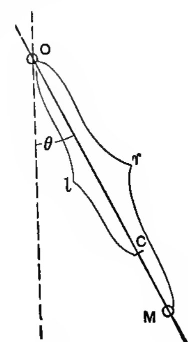


FIG. 67

Then by the results of Section XIX the time of one oscillation of this ideal pendulum, consisting of a particle M connected to the axis by a massless rod of length r, is

$$T=\pi\sqrt{\frac{r}{g}};$$

but as $$r=\frac{k^2}{l} \text{ by hypothesis,}$$

$$\therefore\ T=\pi\sqrt{\frac{k^2}{gl}},$$

which is the period of the original compound pendulum.

The point M (Fig. 67) at which the whole mass of the compound pendulum can be concentrated without effecting a change in the period of oscillation is called the *Center of Oscillation* of the pendulum and the point O on the axis of suspension is called the *Center of Suspension.*

EXERCISE 267. A compound pendulum has a weight of 64 pounds and a principal radius of gyration of 4 feet; its centroid is at a distance of 5 feet from the axis. Find the distance between the center of oscillation and the center of suspension, and the time of oscillation for this pendulum.

EXERCISE 268. Show that a thin rigid wire will oscillate with the same period whether it be suspended from its extremity or from a point one-third of the length from its extremity.

The *centers of suspension and oscillation are interchangeable.* Thus the time of oscillation will be unal-

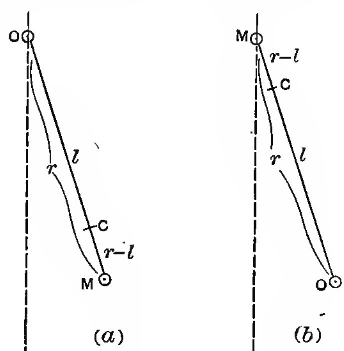


FIG. 68

tered if the pendulum be suspended from its center of oscillation and the old center of suspension be considered as the new center of oscillation.

This is illustrated in Fig. 68.

In order to prove the above theorem we must first find the relation between the principal radius of gyration, $\bar{k}$, of the pendulum and l and r.

Thus if m is the mass of the pendulum and as

$$I_0 = I_c + l^2 m,$$

we have $$k^2 m = \bar{k}^2 m + l^2 m,$$

or $$k^2 = \bar{k}^2 + l^2.$$

But $$r = \frac{k^2}{l},$$

$$\therefore\ \bar{k}^2 = l(r-l).$$

If k_1 be the radius of gyration of the pendulum about an axis through M, then the time of oscillation for Fig. 68 (b) would be

$$T_b = \pi\sqrt{\frac{k_1^2}{(r-l)g}}.$$

But $$k_1^2 = \bar{k}^2 + (r-l)^2 = l(r-l) + (r-l)^2 = (r-l)r,$$

$$\therefore\ T_b = \pi\sqrt{\frac{r(r-l)}{(r-l)g}} = \pi\sqrt{\frac{r}{g}}.$$

As the time of oscillation of the pendulum shown in Fig. 68 (a) is

$$T_a = \pi\sqrt{\frac{k^2}{lg}} = \pi\sqrt{\frac{r}{g}},$$

$$T_a = T_b,$$

or the time of oscillation of the pendulum in its original position equals the time of oscillation in its reversed position.

The principle just proved was made use of by Captain Kater in 1818 for determining the length of the equivalent simple pendulum for a given compound pendulum, and thus obtaining the value of the acceleration of gravity, g. (See your text-book of Physics.)

Experimental Determination of the Radius of Gyration of a Given Body

In the practice of engineering it is often important to find the moment of inertia (or radius of gyration) of a moving part of a machine, say the connecting-rod of a steam-engine. On page 155 we deduced the fact that

$$\bar{k}^2 = l(r-l),$$

where $\bar{k}$ is the radius of gyration of the body about its centroid, l the distance from the point of suspension to the centroid, and $r-l$ the distance from the point of oscillation to the centroid. This equation enables us to find the required moment of inertia experimentally.

In Fig. 69 is shown the connecting-rod whose moment of inertia is required. Let C be its centroid and O_a and O_b the centers of oscillation when suspended from A and B, respectively.

To find r_a experimentally, suspend the connecting-rod from A and determine the time of oscillation. Then by means of

$$T = \pi\sqrt{\frac{r_a}{g}},$$

find r_a. By suspension from B, r_b may be similarly found.

Note that r_a and r_b are the lengths of equivalent simple pendulums.

To find l_a, support the rod on knife-edges at A and B and weigh the pressures on the knife-edges. Then by

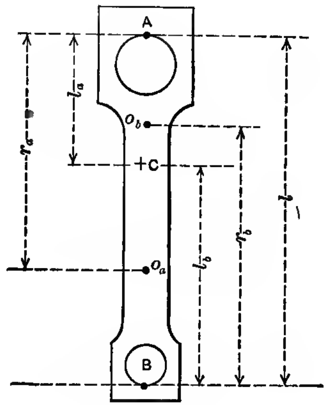


FIG. 69

the principles of Statics, if W_a and W_b are the respective components of the weight W of the bar, we have

$$l_a=\frac{W_b}{W}l \quad \text{and} \quad l_b=\frac{W_a}{W}l.$$

Thus $$\bar{k}^2=l_a(r_a-l_a) \quad \text{or} \quad \bar{k}^2=l_b(r_b-l_b),$$

either of which gives the required value and one serves to check the other.

EXERCISE 269. State clearly the meaning of $\bar{k}$ as used above. If the weight of the connecting-rod used above be W, find its moments of inertia about A and B as axes in terms of $\bar{k}$. What would be the moment of inertia about an axis through the center of the hole at A if the radius of this hole is x?

EXERCISE 270. The following data were obtained from a connecting-rod whose principal radius of gyration was to be found:

Length of rod between extremes of holes 52.8 inches.

Weight on knife-edge under extreme of large hole, 54.5 pounds.

Weight on knife-edge under extreme of small hole, 50.65 pounds.

Time of 200 vibrations, large end down, $3{:}26\frac{1}{5}$ minutes.

Time of 200 vibrations, small end down, $3{:}24\frac{2}{5}$ minutes.

Find $\bar{k}$.

SECTION XXIV

ROTATION PRODUCED BY VARIABLE FORCES

The Torsion Balance.—If a body be suspended by an elastic wire tightly clamped at its upper end in such a manner as to allow an angular displacement about the wire as an axis, the combination is known as a torsion balance.

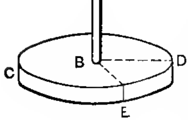


FIG. 70

In Fig. 70, let AB be the elastic wire, and the disk BD the suspended body. If D be displaced through a certain angle so that D moves to E, the wire AB will be twisted, and by reason of its elasticity will tend to return to its initial condition, and if the disk be released it will return to, and swing beyond, its initial position. Experiment shows that the moment or torque exerted by a wire, twisted as just

described, is proportional to the angular displacement of the disk.

Example.—Find the time of oscillation of the disk shown in Fig. 70, if the moment of inertia of the disk about the wire as axis is I, and the wire exerts a moment M_1 when the disk is displaced through an angle θ_1.

Solution.—Let Fig. 71 represent a top view of the disk. Assume D as the position of a mark on the disk when at rest.

Assume the disk to be displaced through an angle θ_0, so that the mark moves to E, and then be released when

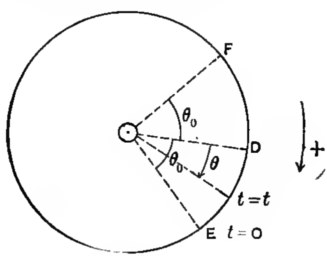


FIG. 71

$t=0$. Let θ be the displacement of the disk at the time t. Then at that instant the wire exerts a torque M upon the disk and $M \propto \theta$ or $M=c\theta$, but when $M=M_1$, $\theta=\theta_1$, so that $c=\dfrac{M_1}{\theta_1}$.

By the equation of motion for rotation we have

$$-c\theta=I\alpha=I\frac{\omega\, d\omega}{d\theta},$$

where ω is the angular velocity of the disk and the moment $c\theta$ is made negative, as it tends to turn the disk in the negative direction.

Integrating this equation and multiplying by 2 we obtain

$$\omega^2 I = -c\theta^2 + C_1.$$

As $\omega = 0$ when $\theta = \theta_0$, $C_1 = c\theta_0^2$.

$$\therefore\ \omega^2 I = c(\theta_0^2 - \theta^2),$$

and $$\omega = \frac{d\theta}{dt} = \sqrt{\frac{c}{I}}\sqrt{\theta_0^2 - \theta^2}. \quad . \quad . \quad . \quad . \quad (1)$$

Integrating, we have

$$t = \sqrt{\frac{I}{c}} \sin^{-1}\frac{\theta}{\theta_0} + C_2.$$

As $\theta = \theta_0$ when $t = 0$, $C_2 = -\sqrt{\frac{I}{c}}\frac{\pi}{2}$.

$$\therefore\ t = \sqrt{\frac{I}{c}}\left\{\sin^{-1}\frac{\theta}{\theta_0} - \frac{\pi}{2}\right\}, \quad . \quad . \quad . \quad . \quad (2)$$

where t is the time occupied in turning from E to any displacement indicated by the angle θ.

Now make $\theta = 0$, then $\sin^{-1}\frac{\theta}{\theta_0} = 0, \pi$, etc.; taking the value π,

$$t = \frac{\pi}{2}\sqrt{\frac{I}{c}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the time in which the disk rotates from E to D.

In equation (1)

$$\omega = \sqrt{\frac{c}{I}}\sqrt{\theta_0^2 - \theta^2}.$$

To find the displacement of the disk when it is instantaneously at rest put $\omega = 0$ and solve for θ.

Thus $\theta = \pm\theta_0$, so that the disk is at rest when the mark is at E and at F, equidistant from D.

Thus the value of t in (3) is the time for a quarter period or the period of a complete oscillation is

$$T = 2\pi\sqrt{\frac{I}{c}} = 2\pi\sqrt{\frac{I\theta_1}{M_1}}.$$

EXERCISE 271. A disk, radius 3 feet and weight 640 pounds, is suspended as the pan of a torsion balance by a wire whose torsional strength is determined by noting that forces of 100 pounds applied to opposite ends of a diameter of the disk displace the disk through $\frac{1}{2}$ a radian. If the disk be displaced through 2 radians, find the time of one oscillation.

EXERCISE 272. What would be the time of oscillation of a sphere of the same weight as the disk in Ex. 271 suspended by the same wire, the radius of the sphere being 3 feet?

Experimental Determination of Moments of Inertia by Means of the Torsion Balance

It has been shown that the time of a complete oscillation of a torsion balance is $T = 2\pi\sqrt{\frac{I}{c}}$, where I is the moment of inertia of the balance about the suspending wire as axis and c a constant depending upon the elastic properties of the suspending wire.

For our present purpose the above equation may be written

$$T = \frac{2\pi}{\sqrt{c}}\sqrt{I} = N\sqrt{I},$$

where N is a constant for a given wire.

In the experimental determination of moments of inertia by the principle of the torsion balance the frame shown in Fig. 72 is used. On the upper board of the frame two equal weights, W, are placed always in such a position that their centroid is directly below the wire. On the lower board the body whose moment of inertia is to be determined is placed with its centroid also directly below the wire. (To place the weights and body in the proper position, shift them about until the frame is level.)

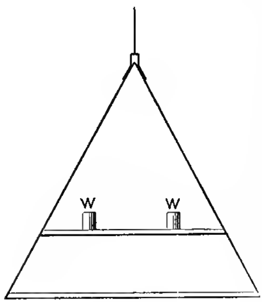


FIG. 72

As the moment of inertia of the frame is unknown, we must first make an experiment to eliminate it.

(1) Place the weights, W, so that their centroids are at a distance d_1 feet apart and determine the time of one oscillation, say T_1. Then $T_1 = N\sqrt{I_1}$, where I_1 is the moment of inertia of the frame and weights in their first position.

(2) Now shift the weights so that d_2 is the distance between their centroids and determine T_2 experimentally. Then $T_2 = N\sqrt{I_2}$, where I_2 is the moment of inertia of the frame plus the weights in their second position.

If we consider the weights as concentrated at their centroids, then their moment of inertia in (1) is $\dfrac{Wd_1^2}{2g}$ and in (2) is $\dfrac{Wd_2^2}{2g}$, so that the moment of inertia of the

frame and weights in (2) is increased by $\frac{Wd_2^2}{2g}-\frac{Wd_1^2}{2g}$ over its value in (1).

Therefore $$I_2=I_1+\frac{Wd_2^2}{2g}-\frac{Wd_1^2}{2g},$$

and as $$I_2=\frac{T_2^2}{N^2} \quad \text{and} \quad N^2=\frac{T_1^2}{I_1}$$

we have $$\frac{T_2^2}{T_1^2}I_1=I_1+\frac{Wd_2^2}{2g}-\frac{Wd_1^2}{2g},$$

or $$I_1=\frac{T_1^2}{T_2^2-T_1^2}\cdot\frac{W(d_2^2-d_1^2)}{2g}.$$

Thus the moment of inertia of the frame and the weights in the first position (their centroids being d_1 feet apart) has been determined. Now place the body whose moment of inertia is sought upon the lower board of the frame and the weights W in their first position and find the time of one oscillation from which I', the moment of inertia of the frame, weights, and body, can be found.

Then $I'-I_1$ is the moment of inertia of the body.

Section XXV

PLANE MOTION, TRANSLATION AND ROTATION

In Kinematics it has already been shown that any plane motion can be regarded as a rotation about any point plus a translation.

As a free body will always rotate about its mass-center it is both convenient and natural to consider a line,

through the mass-center of the body and perpendicular to its plane of motion, as the axis of rotation.

In solving problems relating to the kinetics of plane motion always consider the body as free and revolving about its mass-center. Then employing the axial components of translation of the mass-center, a_x and a_y, and the angular acceleration about the mass-center, α, write the kinetic equations of translation

$$\Sigma \mathbf{F}_x = \mathbf{a}_x \Sigma \mathbf{m},$$
$$\Sigma \mathbf{F}_y = \mathbf{a}_y \Sigma \mathbf{m},$$

and of rotation

$$\Sigma \mathbf{Moments} = \alpha \mathbf{I}.$$

Then determine the kinematic equations, showing the relation between a_x, a_y, and α, which are necessary to correctly represent the actual motion of the body. These equations are then to be solved for the quantities sought.

Example.—A homogeneous cylinder (radius r) rolls down an inclined plane (inclination β, length l); find the time required to reach the foot.

Solution.—Fig. 73 illustrates the problem and Fig. 74 shows the cylinder as a free body. As the cylinder *rolls* enough friction must be introduced to prevent slipping; this friction need not, however, be the limiting friction.

The acceleration of the mass-center will in this example be parallel to the inclined plane, so assume the axes as shown in Fig. 73.

The equations of motion for translation then are

$$N - W \cos \beta = m a_y$$

and

$$-F + W \sin \beta = m a_x,$$

where m is the mass of the cylinder.

For rotation about the mass-center, C, the equation of motion is

$$Fr = I\alpha,$$

where I is the moment of inertia of the cylinder *about its axis.* These equations are the kinetic equations of the problem.

To obtain the kinematic equations, note that as the point A is the instantaneous center of rotation it can have no velocity along the x-axis and therefore no component

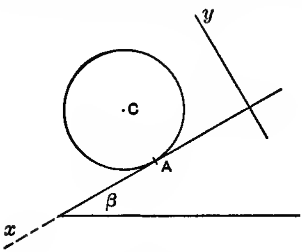


FIG. 73

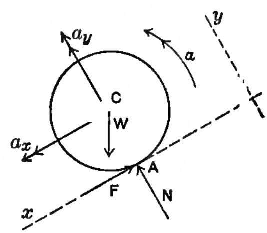


FIG. 74

acceleration along this line. Still, owing to α the point A has an acceleration along the x-axis of $-\alpha r$ combined with a_x due to translation. Therefore

$$a_x = \alpha r,$$

also $$a_y = 0.$$

These are the kinematic equations.

The solution of the kinetic and kinematic equations gives

$$\alpha = \frac{Fr}{I}, \text{ and as } I = \tfrac{1}{2}mr^2, \ \alpha = \frac{2F}{mr};$$

also $$F = \tfrac{1}{3}mg \sin \beta.$$

So that $\alpha = \frac{2}{3}\left(\frac{g \sin \beta}{r}\right)$ and $a_x = \frac{2}{3}g \sin \beta$.

To find the time required to reach the foot of the plane, note that the mass-center moves through l feet with an acceleration of $\frac{2}{3}g \sin \beta$ ft.-per-sec. per sec. Therefore by means of

$$s = \tfrac{1}{2}at^2 + v_0 t$$

we obtain
$$l = \tfrac{1}{3}g \sin \beta \cdot t^2.$$

$$\therefore \ t = \sqrt{\frac{3l}{g \sin \beta}} \text{ seconds.}$$

EXERCISE 273. What would be the linear velocity of the mass-center and the angular velocity of the cylinder in the preceding example at the foot of the plane?

EXERCISE 274. Compare the accelerations of the mass-centers of two cylinders, one rolling and the other sliding (no friction) down an inclined plane.

EXERCISE 275. Find the greatest inclination, β, of the plane in the preceding example consistent with no slipping if μ is the coefficient of friction.

EXERCISE 276. Compare the time of descent on an inclined plane for two spheres, one solid and one hollow, having the same mass, m, and the same external radius, r, while the thickness of the shell is $\frac{r}{2}$.

CHAPTER VIII

WORK AND ENERGY

In Fig. 75 assume the mass m to be acted on by the force F and to possess at a certain time a velocity v_1, and that after having traversed a distance s, its velocity (by reason of the action of the force F) has been increased to v_2.

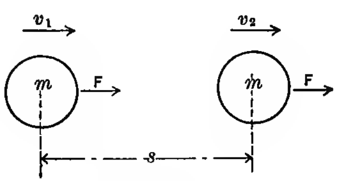


Fig. 75

Using the kinematical formula $v^2 = 2as + v_0^2$, we have $v_2^2 = 2as + v_1^2$, where a is the acceleration produced by the force F in the mass m and therefore $F = ma$.

Substituting for a its value $\frac{v_2^2 - v_1^2}{2s}$, we obtain

$$F = m\frac{v_2^2 - v_1^2}{2s},$$

or

$$Fs = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}.$$

Fs is now defined as the work done by the force F while acting through the space s.

$\frac{mv^2}{2}$ *is defined as the Kinetic Energy of a mass m moving with a velocity v.*

The formula last deduced may thus be interpreted as follows:

The work done by a force is equal to the increase of the kinetic energy of the mass upon which it acts.

SECTION XXVI

WORK

Whenever a body moves under the action of a force, work is said to be done by this force.

The work done is measured by the product of the force and the effective displacement of its point of application.

By *effective displacement* is meant the projection of the displacement upon the line of action of the force.

If the force acting is F and the effective displacement is s, then the work done is

$$\mathbf{Work = Fs.}$$

The *unit of work* is obtained by making $F = 1$ pound and $s = 1$ foot; then

$$\text{Unit of work} = (1 \text{ pound})(1 \text{ foot}) = (1 \text{ foot-pound}).$$

The foot-pound is the work done whenever the point of application of a force of one pound receives an effective displacement of one foot.

It is convenient to speak of positive or negative work. Work is considered *positive* or negative as the effective

displacement *agrees* or is opposed to the force in direction.

EXERCISE 277. A body weighing 100 pounds rests upon a smooth horizontal plane. What work is done in moving the body 10 feet in any direction along the plane? If the plane be rough and the coefficient of friction be 0.5, what would be the work done? Has the time it takes to move the body any effect upon the work done?

EXERCISE 278. A man weighing 140 pounds carries a load of 100 pounds up a ladder 50 feet long, inclined at 60° to the horizontal; how much work does he do?

From Fig. 76, where F is a force, s the displacement, and θ their mutual inclination, we see that

$$\text{Work} = F(s \cos \theta),$$

$$\therefore \text{Work} = s(F \cos \theta);$$

or, *the work done by a force may also be measured by the product of the displacement and the component of the force along the line of displacement.*

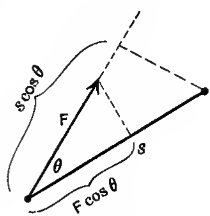


FIG. 76

Example.—Find the work done if a body weighing W pounds is dragged up an inclined plane, whose inclination is θ and whose length is s, with an acceleration a, the coefficient of friction being μ.

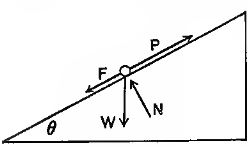


FIG. 77

Solution.—Fig. 77 illustrates the problem and shows *all the forces* acting upon the body. P is the pull necessary to drag the body, N the normal reaction of the plane, F the friction force, and W the weight of the body. In order to calculate the work done it will be

necessary to find P in terms of the known quantities W, θ, μ, and a. This is done by means of the kinetic equations for translation. Thus

$$P - F - W \sin \theta = \frac{W}{g} a,$$

$$N - W \cos \theta = \frac{W}{g}(0) = 0.$$

As $F = \mu N$, $F = \mu W \cos \theta$ and

$$P = \frac{W}{g} a + W \sin \theta + \mu W \cos \theta.$$

Now as Work = (displacement) (effective component of the force)

we have Work done $= (s)(P)$

$$= sW\left(\frac{a}{g} + \sin \theta + \mu \cos \theta\right).$$

Exercise 279. Deduce the dimensions of work. Check by the theory of dimensions the results of the preceding example.

Exercise 280. Find the work done if a body weighing W pounds is dragged down a plane whose length is l, base b, and height h. The coefficient of friction is μ and the velocity is constant.

Exercise 281. A horse hauling a wagon exerts a constant pull of 100 pounds and travels at the rate of 2 miles an hour. How much work will be done in 5 minutes?

Exercise 282. Assume three particles weighing W, U, and V pounds to be at the distances x, y, and z above a horizontal plane. Let each particle be raised through a distance a, b, and c, respectively. Show that the work so done equals the product of the sum of the weights of the particles by the distance their common center of gravity is raised.

The last exercise shows that the *work done in raising a body* (composed of many particles) *may be found by taking the product of the distance through which the C. of G. of the particles has been displaced vertically and the combined weight of these particles.*

EXERCISE 283. A pit 10 feet deep and with a cross-section of 4 square feet is to be excavated and the earth thrown into carts to a height of 4 feet above the ground. How much work must be done, supposing a cubic foot of earth to weigh 90 pounds?

EXERCISE 284. Find the work done in lifting a chain which hangs vertically, its length being l feet and its weight W pounds.

Power or Activity

In the calculating of work done no account is taken of the time required to do this work. It is, however, often of great importance to know and bring into calculation the factor of time. This is especially so if we wish to compare the agents which do the work.

Thus a man can do the same work as may be done by a horse *provided* enough time is allowed the man.

To bring this new view into calculation we need to know the rate at which various agents perform work, i.e., the *foot-pounds of work they can do per second.*

The terms *Power* or *Activity* are used in this connection. Thus we say that the power of a horse is greater than that of a man because a horse can do more work *per second* than a man.

Engineers do not reckon power in *foot-pounds per second,* as it is too small a unit. They employ instead

the *Horse-power,* which equals *550 foot-pounds per second* or *33,000 foot-pounds per minute.*

It should be carefully noted that Horse-power (H.P.) does *not* express an amount of work but a rate of doing work.

EXERCISE 285. Deduce the dimensions of Work, Energy, Power. Which of these quantities can be measured in the same units?

EXERCISE 286. What is the H.P. of an engine that can raise every minute and a half 500 cu. ft. of water to a height of 100 feet (1 cu. ft. of water weighs 62.5 pounds).

EXERCISE 287. How many gallons of water would be raised per hour from a mine 600 feet deep by an engine of 175 H.P., supposing a gallon of water to weigh $8\frac{1}{3}$ pounds?

EXERCISE 288. Find the H.P. necessary to pump out the St. Mary's Falls Canal lock, Sault Ste. Marie, in 24 hours, the length of the lock being 500 feet, width 80 feet, and the depth of water 18 feet, the water being delivered to a height of 42 feet above the bottom of the lock.

Work Done by Variable Forces

If a force varies in direction or magnitude or both, we can still measure the work done by means of the product of the force and the effective displacement, provided the displacement considered is sufficiently small. We then put

$$d(\text{Work}) = Fds \cos \theta;$$

this must be expressed in terms of a single variable and integrated to obtain the total work done.

Example.—A spring whose natural length is l feet is stretched to a length $l+s_1$. If p pounds will stretch the

spring one foot, find the work done by the stretching force.

Solution.—In this example the stretching force evidently varies; when the spring has stretched s feet, the force must be ps, by Hooke's Law. The effective displacement for this force is ds;

$$\therefore \text{ Work done} = \int ps\, ds = \frac{ps^2}{2} + C.$$

Now when $s=0$, the work done is zero, therefore $C=0$, and the work to stretch the spring s feet is $\frac{ps^2}{2}$. The total work done in stretching the spring s_1 feet is $\frac{ps_1^2}{2}$.

EXERCISE 289. When the result of the preceding example is written in the form $\left(\frac{ps_1}{2}\right)(s_1)$, what simpler solution of the problem does it suggest?

EXERCISE 290. The wire for moving a distant signal is stretched 16 inches beyond its natural length and has a tension of 240 pounds when the signal is down. This tension is produced by a back weight of 270 pounds resting with a portion of its weight (30 pounds) upon its bed. If the signal end of the wire is moved through 2 inches in raising the signal, show that the end attached to the hand-lever must move 4 inches. Find the work done when the hand-lever is suddenly pulled back and locked before the signal begins to move, and find how much less work is necessary if it be pulled back slowly.

Graphical Representation of Work

It is often convenient, especially when the forces vary, to represent graphically the work done. In Fig. 78 let the horizontal axis be the axis of effective displacement and the vertical axis the axis of force. Then during the displacement $s_2 - s_1$ the *work diagram* shows the action of a constant force F_1 and the work done is represented by the area of the shaded rectangle. (Why?) During

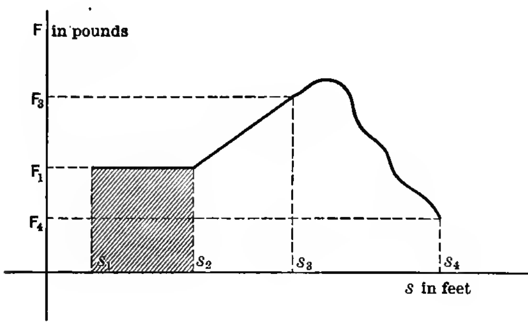


FIG. 78

the displacement $s_3 - s_2$ the work diagram shows the action of a uniformly increasing force from F_1 to F_3 and evidently the work done is represented by the area of the trapezoid whose side is $(s_2 s_3)$. During the displacement $s_4 - s_3$ a varying force acts, but the area between the s-axis, the curve, and the ordinates at s_3 and s_4 still represents the work done.

EXERCISE 291. Draw the work diagrams for Exs. 289 and 290.

SECTION XXVII

ENERGY

As explained on page 167, a force acting on a mass increases its velocity and therefore its kinetic energy if the force acts in the direction of the velocity. Similarly, a force acting in the opposite direction would decrease the kinetic energy of the mass. In this case the mass is said to do work in opposing the force.

Thus a moving mass has the ability to do work. This ability to do work is called *Energy.*

Bodies may possess this ability to do work even though not in motion. Thus a weight above the ground possesses energy, for if allowed to descend it could do work.

Kinetic Energy is energy due to motion. *Potential Energy* is energy due to position.

The kinetic energy of a particle is defined as

$$\frac{(\text{mass})(\text{velocity})^2}{2}.$$

The unit of energy must be the same as the unit of work, for energy is simply stored work.

EXERCISE 292. A particle weighing 64 pounds and moving east with a velocity of 3 ft. per sec. receives a blow such that the velocity due to it is 4 ft. per sec. north. Find its kinetic energy before and after it received the blow.

EXERCISE 293. A ball weighing one pound is fired vertically upward from the ground with a velocity of 160 ft. per sec. Compute and tabulate its kinetic and potential energy at the start and at the end of each second during the time it ascends.

This exercise illustrates the principle of the *transformation of energy* and also the principle of the *conservation of energy.*

Kinetic Energy of a Body

As the kinetic energy of a particle is $\frac{mv^2}{2}$, and as a body can be conceived as a system of particles, we have, for the kinetic energy of a

TRANSLATING BODY

whose velocity is v, $\frac{1}{2}\Sigma mv^2 = \frac{1}{2}v^2\Sigma m$

$$= \frac{\mathbf{M}\mathbf{v}^2}{\mathbf{2}},$$

where M is the mass of the whole body.

ROTATING BODY

Let ω be the angular velocity of the rotating body, then the linear velocity of any particle whose distance from the axis of rotation is r will be ωr. Thus the kinetic energy of this particle will be $\frac{m\omega^2 r^2}{2}$, and the kinetic energy of the whole body will be $\frac{1}{2}\Sigma m\omega^2 r^2 = \frac{\omega^2}{2}\Sigma mr^2$, but Σmr^2 is the moment of inertia of the body, so that the kinetic energy of rotation is

$$\frac{\mathbf{I}\omega^2}{\mathbf{2}}.$$

BODY WITH ANY PLANE MOTION

If we consider the instantaneous axis of a body as its axis of rotation, the kinetic energy it possesses would be

$\frac{I\omega^2}{2}$, where I is the moment of inertia about its instantaneous axis and ω its angular velocity about the same axis.

The instantaneous axis, however, continually changes its position. It is therefore more convenient to consider the kinetic energy independently of it.

Let $\bar{v}$ be the velocity of the mass-center, $\bar{r}$ its distance from the instantaneous axis, $\bar{I}$ the moment of inertia of the body about an axis through the mass-center and perpendicular to the plane of motion, and M the mass of the body; then

$$I=\bar{I}+M\bar{r}^2 \quad \text{and} \quad \bar{v}=\bar{r}\omega,$$

$$\therefore \frac{I\omega^2}{2}=\tfrac{1}{2}\{\bar{I}\omega^2+M\bar{r}^2\omega^2\},$$

or the kinetic energy is

$$\frac{\bar{\mathbf{I}}\omega^2}{2}+\frac{\mathbf{M}\bar{\mathbf{v}}^2}{2},$$

where ω is the angular velocity about the mass-center.

The kinetic energy of a body consists of two parts, one due to the translation of the mass-center and the other due to the rotation about the mass-center.

Exercise 294. Find the kinetic energy of a cylindrical plate (radius 2 feet and weighing 394 pounds) when

(*a*) making 240 revolutions per minute about its axis,

(*b*) rolling along horizontal ground with a velocity of 7 ft. per sec.,

(*c*) sliding with a velocity of 7 ft. per sec.

EXERCISE 295. A coin rolls on its edge in a vertical plane. Compare its rotational and total energies.

EXERCISE 296. The weight of a fly-wheel is W pounds, the mean diameter of the rim is d feet, and the number of revolutions per second is n. Find the energy stored in the moving wheel.

EXERCISE 297. Show that the energy of a sphere when rolling without sliding is $\frac{7}{5}$ of its energy when sliding without rolling for the same velocity of its mass-center.

EXERCISE 298. How much more energy is stored per ton in a car-wheel than in a car-body if the car travels at a velocity of 30 miles per hour, the wheel being 28 inches in diameter?

SECTION XXVIII

PRINCIPLE OF WORK

On page 167 it was shown that the work done by a force is equal to the increase of kinetic energy. If several forces act, the same law holds, provided due attention is given to the signs of the various portions of work done by each force and these are then summed up.

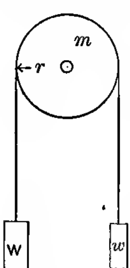


FIG. 79

The principle of work can then be stated as follows:

Total Work Done by the Forces = Change in Kinetic Energy.

Example.—Find by the principle of work the velocity, v, attained by the weights of an Atwood machine after they have moved s feet from rest.

Solution.—As in Fig. 79, let the weights be W and w, let the mass of the pulley be m, its radius r, and its principal radius of gyration k. Neglecting

friction, the *work done* during a displacement, s, of the weights is

$$Ws - ws,$$

and the change in *kinetic energy* is

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 + \frac{1}{2}\left(\frac{w}{g}\right)v^2 + \tfrac{1}{2}(mk^2)\left(\frac{v}{r}\right)^2;$$

$$\therefore\ (W - w)s = \tfrac{1}{2}v^2\left\{\frac{W+w}{g} + \frac{mk^2}{r^2}\right\},$$

or

$$v^2 = 2gs\left\{\frac{(W-w)r^2}{(W+w)r^2 + mgk^2}\right\}.$$

Example.—Find the depth, x, through which a pile can be driven by a blow delivered by a ram weighing W pounds and falling from a height h if R is the effective resistance of the ground.

Solution.—In this case the work done on the ram by gravity is Wh, the work done against the resistance of the ground is $-Rx$, and the change in kinetic energy is zero.

$$\therefore\ Wh - Rx = 0 \quad \text{or} \quad x = \frac{Wh}{R}.$$

EXERCISE 299. A slider weighing 100 pounds rests on a table; it is moved by a weight of 20 pounds fastened to it by a cord which passes over a pulley at the edge of the table. When the slider has moved 2 feet its velocity is 2 ft. per sec. Find the coefficient of friction.

EXERCISE 300. Solve Ex. 183, page 108, by the principle of work.

EXERCISE 301. Solve Ex. 227, page 132, by the principle of work.

Exercise 302. A disk and a hoop roll with the same velocity on the level and commence to ascend an incline. If they have the same mass and radius, which will ascend higher and by how much?

Exercise 303. A bullet weighing one ounce leaves the mouth of a rifle, whose barrel is 4 feet long, with a velocity of 1000 ft. per sec. Find the mean pressure on the bullet, neglecting friction.

Exercise 304. The head of a steam-hammer weighs 10 tons and has a fall of 8 feet. If it indents the iron on which it falls one inch, find the mean force exerted on the iron during compression.

Exercise 305. A mass of 10 pounds slides along a horizontal surface with a velocity of 20 ft. per sec. when it strikes a spring. It requires 2 pounds to compress the spring one inch. If the coefficient of friction is 0.1, how far will the mass move after striking the spring before coming to rest, and what potential energy is stored in the spring when the body comes to rest?

Exercise 306. A locomotive draws a load of 200 tons. Find the draw-bar pull (*a*) at constant speed if the friction is 0.05 of the load; (*b*) if the friction is the same and the velocity increases from 30 ft. per sec. to 40 ft. per sec. while moving one mile.

Exercise 307. A train weighing 60 tons has a velocity of 40 miles per hour when the power is shut off. If the resistance to motion is 10 pounds per ton, how far will the train move before the velocity reduces to 10 miles per hour?

Exercise 308. A fly-wheel weighing 15 tons has a mean diameter of 20 feet and makes 60 revolutions per minute. If the axle of the wheel is 14 inches in diameter and the coefficient of friction is 0.2, how many revolutions will the wheel make before coming to rest?

SECTION XXIX

APPLICATION TO MACHINES

The principle of work as discussed in Section XXVIII can be used advantageously to calculate the forces acting on a machine.

Example.—Find the force necessary to raise a weight W by means of a wheel and axle if the coefficient of friction is μ.

Solution.—Let the radii of the wheel and axle be R and r, and the radius of the journal r'. The friction force acting at the surface of the journal will be $\mu(F+W)$, where F is the force sought. If the displacement considered is one revolution, then the work done is

$$F(2\pi R)-W(2\pi r)-\mu(F+W)(2\pi r').$$

This must equal zero, as we assume no change in the velocity.

$$\therefore\ FR-Wr-\mu(F+W)r'=0,$$

or

$$F=\frac{Wr+\mu Wr'}{R-\mu r'}.$$

EXERCISE 309. Find the force necessary to raise W pounds by means of a single fixed pulley if the radius of the pulley is r, that of the journal r', and the coefficient of friction is μ.

EXERCISE 310. Find the mechanical advantage of a differential wheel and axle, neglecting friction (Fig. 80).

EXERCISE 311. What force would be required to raise 10 cubic feet of iron (specific gravity 7) when under water by means of the machine illustrated in Fig. 80?

The Screw.—Consider a screw-jack with a moving force F applied at the end of a lever l feet long and about to raise a weight W, the coefficient of friction between the nut and the screw being μ. Let the pitch of the threads be p and the mean radius of the screw be r. Consider the screw unwrapped, as shown in Fig. 81; the thread then forms the hypothenuse of a triangle. The sum of the normal pressures between the screw and the

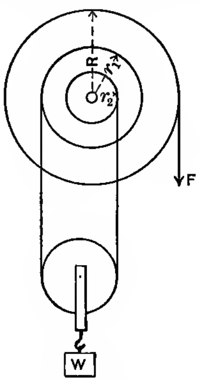


FIG. 80

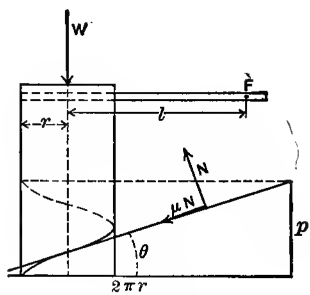


FIG. 81

nut, N, and the sum of the friction forces, μN, act as shown.

Let the displacement be one revolution of the screw, then the force F will be displaced $2\pi l$ feet, the weight, W, $p=2\pi r \tan\theta$ feet, the friction forces, μN, will be displaced along one turn of the screw, or the hypothenuse of the unwrapped screw $2\pi r \sec\theta$, while the normal pressure suffers no effective displacement. Therefore by the principle of work, assuming no change in velocity, we have

$$F(2\pi l) - W(2\pi r \tan\theta) - \mu N(2\pi r \sec\theta) = 0.$$

We must now determine N. This can be done by placing the sum of the vertical components of all forces equal to zero, as there is no acceleration; thus

$$-W+N\cos\theta-\mu N\sin\theta=0.$$

$$\therefore\ N=\frac{W}{\cos\theta-\mu\sin\theta},$$

and $$Fl=Wr\tan\theta+\frac{\mu Wr\sec\theta}{\cos\theta-\mu\sin\theta}.$$

If ϕ be the angle of friction so that $\mu=\tan\phi$, this expression reduces to

$$Fl=Wr\tan(\theta+\phi).$$

EXERCISE 312. Show that if the weight W in the preceding discussion is about to move *down* the force F_1 applied at the lever is

$$F_1=\frac{Wr}{l}\tan(\theta-\phi).$$

EXERCISE 313. If $\theta=\phi$, interpret the result of Ex. 312.

EXERCISE 314. The pitch of a screw in a screw-jack is $\frac{7}{16}$ inch, it is turned by a handle 19 inches long, the depth of the thread is $\frac{19}{40}$ of the pitch, the diameter of the cylinder is 3 inches. If the coefficient of friction is 0.06, what force is necessary to raise 200 pounds?

The *Efficiency* of a machine is the ratio of the useful work to the total work, or

$$\textbf{Efficiency}=\frac{\textbf{useful work}}{\textbf{total work}}.$$

EXERCISE 315. Find the efficiencies of the machines described in Ex. 309, 310, and 314.

The *Mechanical Advantage* or *force ratio* of a machine is the ratio of the force overcome to the force exerted, or

$$\textbf{Mechanical advantage} = \frac{\textbf{force overcome}}{\textbf{force exerted}}.$$

EXERCISE 316. Find the mechanical advantages of the machines described in Ex. 309, 310, and 314.

In certain machines, such as the steam-engine, where the driving force is the varying pressure of a fluid, an "indicator" is used to determine the pressure at each point of the stroke. The mean effective pressure can thus be found experimentally. The horse-power calculated by means of this mean effective pressure is called the Indicated Horse Power (I.H.P.).

EXERCISE 317. Show that $\frac{2plan}{33,000}$ is the I.H.P. of a double-acting engine if p = mean effective pressure in pounds per square inch, l = length of stroke in feet, a = area of piston in square inches, n = number of revolutions per minute.

EXERCISE 318. The cylinder of a steam-engine has an internal diameter of 3 feet, length of stroke 6 feet, and it makes 10 strokes per minute. What must be the mean effective pressure so that the I.H.P. is 125?

EXERCISE 319. An engine working at 50 H.P. with a mean effective pressure of 75 pounds per square inch in two cylinders has a stroke of 2 feet. If the area of each piston is 72 square inches, how many revolutions will the engine make per minute?

EXERCISE 320. What I.H.P. will be developed by a gas-engine having a 12-inch piston and an 8-inch crank when making 150 revolutions per minute with a mean pressure of 62 pounds per square inch? (One power stroke per revolution.)

The I.H.P. of an engine gives the energy exerted by the steam or fluid used to drive the engine and not the work which the engine can do in overcoming resistances. The commercial value of an engine depends largely on its efficiencv or the ratio of the work done to the energy exerted. One method of calculating the work done is to use an instrument, called a Dynamometer, for measuring the output of work.

One form of dynamometer is shown in Fig. 82. Here P represents a pulley fastened to the shaft of the engine and rotating with it. A rope circles about it and carries at one end a weight W while its other end is fastened to a spring balance.

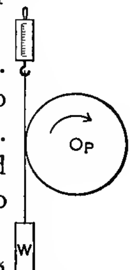


FIG. 82

If the pulley rotates in the direction shown by the arrow the friction between the rope and the pulley will tend to raise the weight W and diminish the reading of the spring balance.

Assume the pulley to rotate with a constant velocity of n revolutions per minute. It will be acted on by three forces: the force exerted by the engine; N, the sum of the normal pressures of the rope; and F, the sum of the friction forces due to the rope. If we denote the work done by the engine per minute by E and the radius of the pulley by r, we have by the principle of work

$$E - F(2\pi rn) = 0,$$

for the normal pressure exerted by the rope suffers no effective displacement,

$$\therefore\ E = 2\pi rnF.$$

Consider now the rope acted on by four forces: N, the sum of the normal pressure exerted by the pulley; T, the pull exerted by the spring balance; F, the sum of the friction forces; and W. As the rope is in equilibrium the sum of the moments about the center of the shaft must be zero. (Why is the center of the shaft selected as the origin of moment?)

Therefore $Wr - Tr - Fr = 0$, the normal pressures not entering the equation, as their lines of action all pass through the center of the shaft. Thus

$$F = W - T,$$

and $$E = 2\pi rn(W - T).$$

Under the conditions assumed the dynamometer absorbs and wastes in friction $\frac{2\pi rn(W-T)}{33{,}000}$ H.P.

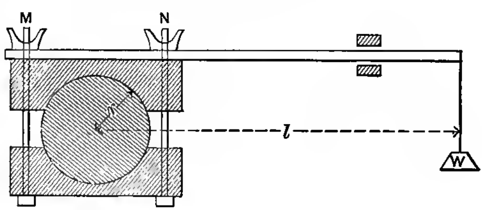


FIG. 83

This result is called the *brake horse-power* (B.H.P.) of the engine and the efficiency of the engine would be the ratio of the B.H.P. to the I.H.P.

EXERCISE 321. In Fig. 82 assume W as 100 pounds and the reading of the spring balance as 120 pounds. In which direction does the shaft turn? What power is absorbed if

the shaft makes 100 revolutions per minute, the radius of the pulley being 2 feet?

Another form of dynamometer is shown in Fig. 83; this form is known as the *Prony Brake.* Here the friction and thus the power absorbed can be varied by tightening the screws M and N.

EXERCISE 322. Calculate the power absorbed by the brake shown in Fig. 83 if a weight W at an arm l just balances the torque due to the friction of the brake and the shaft makes n revolutions per minute.

EXERCISE 323. How many revolutions per minute must the shaft of an engine make if its B.H.P. is 50 and a Prony brake is balanced by 100 pounds at an arm of 4 feet?

CHAPTER IX

IMPACT

SECTION XXX

INTRODUCTION AND DEFINITIONS

IMPACT (or Collision) occurs whenever two bodies moving towards each other come in contact. During contact each body exerts a force upon the other and the velocities of the bodies change rapidly; thus great accelerations are produced and therefore great forces are called into play. This is illustrated in the use of a hammer. If the colliding bodies were rigid, the duration of the impact would be instantaneous and infinite forces would result. If we examine the surface of two ivory balls, previously oiled, after impact, it will be noticed that the area of contact during impact must have been much greater than when they simply rest against each other. During collision the distance between their mass-centers is less than the sum of the radii of the spheres. The mutual forces exerted increase from zero to a maximum, which occurs when the compression is greatest, and then decreases to zero. Thus the duration of impact can be separated into two periods, a period of *compression* and a period of *restitution*. At the instant of maximum compression,

just before restitution takes place, the bodies must possess the same velocity.

It is found by experiment that the velocity of one body relative to the other before impact is never equal to the relative velocity after impact. This is due to the imperfect elasticity of the bodies, owing to which the sum of the forces acting during restitution is always less than the sum of the forces of compression. The ratio of the relative velocity after impact to the relative velocity before impact is called the *coefficient of restitution* and is denoted by e. It is a constant to be determined experimentally. Thus

$$e=-\frac{\text{relative velocity after impact}}{\text{relative velocity before impact}},$$

the minus sign being introduced to render e positive.

If the colliding bodies are perfectly elastic, then $e=1$; if perfectly inelastic $e=0$. In nature neither of these extremes occurs. Some values of e are as follows: glass, 0.94; ivory, 0.81; cast iron, 0.66; lead, 0.2.

When the velocities of the mass-centers are parallel before collision, the impact is called *direct.*

When the line of action of the resultant force of impact passes through the mass-centers of both colliding bodies and the velocities of the mass-centers are along this line, the impact is said to be *direct* and *central.*

If the line of action of the force of impact coincides with the direction of the velocity of the mass-center of one of the bodies but does not pass through the mass-center of the other body, the impact is *direct* and *eccentric.*

If none of the above conditions are fulfilled, the impact is *oblique.*

While considering the impact of bodies, forces other than those due to impact—such as weight, etc.—can be neglected. The reason for this becomes evident when the short duration of the impact and therefore the relatively great magnitude of the forces of impact are taken into account.

EXERCISE 324. What relation exists between the relative velocities before and after impact when the bodies are (*a*) perfectly elastic, (*b*) perfectly inelastic?

EXERCISE 325. Draw sketches illustrating the various kinds of impact.

SECTION XXXI

IMPACT UPON A FIXED SMOOTH PLANE

The fixed plane in this case is to be considered as a portion of the earth, so that the impact really occurs against the whole earth. The mass of the earth being far greater than that of the impinging body, we will consider the velocity of the earth to remain unchanged by the impact or more simply to remain zero.

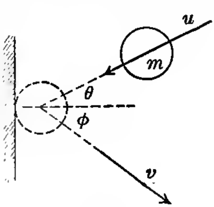


FIG. 84

As in Fig. 84, let the velocity, u, of the sphere before impact make an angle θ with the normal to the plane. Resolve this velocity into a normal ($u \cos \theta$) and a tangential ($u \sin \theta$) component. If friction is neglected, only the normal component will be affected by the impact.

If e is the coefficient of restitution, v the velocity of

the sphere after impact, and ϕ its inclination to the normal,

$$v \sin \phi = u \sin \theta,$$

and
$$v \cos \phi = -e(-u \cos \theta) = e\, u \cos \theta,$$

from which both v and ϕ can be found

EXERCISE 326. Solve for v and ϕ above.

EXERCISE 327. A billiard-ball strikes a cushion at an angle of 45°. If $e = \frac{3}{4}$, find the angle of rebound. What would be the angle if perfect elasticity is assumed?

EXERCISE 328. A ball falls from a height 16 feet above a level floor. Find the velocity of rebound and the height to which the ball will rebound if the coefficient of restitution is 0.75.

EXERCISE 329. At what angle must a body whose coefficient of restitution is $\frac{1}{3}$ be incident on a hard plane that the angle made by its path before and after impact may be a right angle?

EXERCISE 330. A ball, whose coefficient of restitution is e, projected from a given point in a circle returns to the same point after two rebounds from the interior of the circle. Find the angle, θ, made by the direction of projection with the radius at the given point.

SECTION XXXII

DIRECT CENTRAL IMPACT

Consider two spheres (masses m_1 and m_2) moving along their line of centers with velocities u_1 and u_2 before impact (Fig. 85 (a)). During the first period of impact

the variable force of compression, P, acts until the spheres have a common velocity, V (Fig. 85 (b)). Therefore, considering the mass m_1, we have at any instant during compression

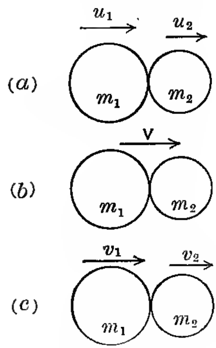


FIG. 85

$$-P = m_1 a = m_1 \frac{dv}{dt},$$

or

$$-\int P dt = m_1 v \Big]_{u_1}^{V} = m_1(V - u_1).$$

From mass m_2,

$$P = m_2 \frac{dv}{dt},$$

or

$$\int P dt = m_2 v \Big]_{u_2}^{V} = m_2(V - u_2).$$

P, although varying, acts equally upon m_1 and m_2, but in opposite directions, and the time during which it acts is evidently equal for both, so that we may equate the values of $\int P dt$, which measures the total *impulse* acting upon either sphere during compression, and obtain

$$m_1(u_1 - V) = m_2(V - u_2),$$

from which $$V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}.$$

Similarly, if R represents the varying force of restitution, we have for the second period of the impact

$$-\int R dt = m_1 v \Big]_{V}^{v_1} = m_1(v_1 - V),$$

and $$\int R dt = m_2 v \Big]_{V}^{v^2} = m_2(v_2 - V);$$

so that $$m_1(V-v_1)=m_2(v_2-V),$$

and $$V=\frac{m_1v_1+m_2v_2}{m_1+m_2}.$$

Equating the values of V, we find

$$\mathbf{m_1u_1+m_2u_2=m_1v_1+m_2v_2}. \quad . \quad . \quad . \quad . \quad (1)$$

This equation contains two unknowns, v_1 and v_2; to solve for either we must have recourse to the experimental fact that

$$\mathbf{e}=-\frac{\mathbf{v_1-v_2}}{\mathbf{u_1-u_2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v_1-v_2 and u_1-u_2 are the velocities of m_1 relative to m_2 after and before the impact.

From the last equation we find

$$v_2=e(u_1-u_2)+v_1,$$

and then from (1),

$$v_1=\frac{m_1u_1+m_2u_2-em_2(u_1-u_2)}{m_1+m_2}.$$

Similarly

$$v_2=\frac{m_1u_1+m_2u_2+em_1(u_1-u_2)}{m_1+m_2}.$$

EXERCISE 331. Find the total impulse exerted during the impact by considering the change of momentum of either mass.

Solve the following exercises without substituting in the results above obtained.

EXERCISE 332. Show that two spheres of equal weight and perfectly elastic will exchange their velocities after impact.

EXERCISE 333. Two inelastic bodies weighing 12 and 7 pounds move in the same direction with velocities of 8 and 5 feet per sec. Find the velocity lost by one and that gained by the other.

EXERCISE 334. A mass m_1 moving with a velocity of 11 ft. per sec. impinges on a mass m_2 moving in the opposite direction with a velocity of 5 ft. per sec. By impact m_1 loses one-third of its momentum. What are the relative magnitudes of m_1 and m_2?

EXERCISE 335. Two bodies of masses m_1 and m_2 are perfectly elastic and move in opposite directions, m_1 is treble m_2, but m_2's velocity is double that of m_1. Determine their velocities after impact.

EXERCISE 336. $m_1(=3m_2)$ impinges on m_2 at rest. m_1's velocity after impact is $\frac{8}{5}$ of its velocity before impact. Find e.

EXERCISE 337. Two bodies m_1 and m_2 moving in opposite directions, with velocities of 25 and 16 ft. per sec., collide. Find the distance between them 4.5 seconds after impact. $(e=\frac{2}{3}.)$

Loss of Kinetic Energy in Impact

During the first period of impact the velocities of the impinging masses change from u_1 and u_2 to a common velocity V.

The kinetic energy after impact is

$$\tfrac{1}{2}(m_1+m_2)V^2=\frac{(m_1u_1+m_2u_2)^2}{2(m_1+m_2)}.$$

Before impact the energy was

$$\tfrac{1}{2}m_1u_1{}^2+\tfrac{1}{2}m_2u_2{}^2=\frac{(m_1{}^2+m_1m_2)u_1{}^2+(m_1m_2+m_2{}^2)u_2{}^2}{2(m_1+m_2)}.$$

Therefore subtracting the kinetic energy after impact from that before impact we have

$$\frac{m_1 m_2(u_1^2+u_2^2-2u_1u_2)}{2(m_1+m_2)}=\frac{m_1m_2(u_1-u_2)^2}{2(m_1+m_2)}.$$

As this is always a positive quantity, the kinetic energy must have decreased. This is to be expected, for the work of compression during the first period is done at the expense of the kinetic energy of the system.

If the bodies are perfectly inelastic none of this loss is returned to the system, as there are no forces of restitution.

If the bodies are perfectly elastic all the energy returns to the system during the period of restitution.

Exercise 338. If the coefficient of restitution is e, show that the loss of kinetic energy of the system is

$$\frac{m_1m_2(1-e^2)(u_1-u_2)^2}{2(m_1+m_2)}.$$

What forms does this energy assume?

Section XXXIII

DIRECT ECCENTRIC IMPACT

This section differs from Section XXXII in that here the line of action of the force of impact no longer passes through the mass-center of both bodies. This new condition will, of course, produce both a translation and a rotation in one of the bodies.

To fix our ideas let the bar (Fig. 86) represent the one body and the sphere the other. Let u_2 be the initial velocity of the sphere, and let its direction coincide with the line of action of the force of impact. Assume the

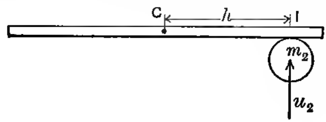


FIG. 86

bar initially at rest; then the motion due to the impact will be a plane motion which can be considered as a rotation about its mass-center, C, and a translation.

Consider only the motion resulting at the end of the

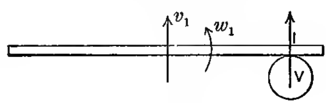


FIG. 87

period of compression and let v_1 represent the velocity of the translation of the bar, ω_1 its angular velocity about C as axis of rotation, and V the common velocity of the

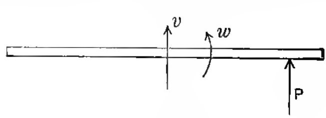


FIG. 88

sphere and the point I of the bar, all at the end of the first period of impact (Fig. 87).

In Fig. 88 the bar is shown as a free body; here P represents the varying force of compression. From this

figure, by reason of the equations of motion for translation and rotation, we have

$$P = m_1 \frac{dv}{dt}$$

and
$$Ph = \bar{k}^2 m_1 \frac{d\omega}{dt},$$

where m_1 is the mass of the bar, $\bar{k}$ its principal radius of gyration, and h the distance between its mass-center, C, and the point of impact, I.

From these equations we obtain

$$\int P dt = m_1 v \Big]_0^{v_1} = m_1 v_1$$

and
$$h \int P dt = \bar{k}^2 m_1 \omega \Big]_0^{\omega_1} = \bar{k}^2 m_1 \omega_1;$$

or
$$v_1 = \frac{1}{m_1} \int P dt, \quad \ldots \ldots \quad (1)$$

and
$$\omega_1 = \frac{h}{\bar{k}^2 m_1} \int P dt = \frac{h}{\bar{k}^2} v_1. \quad \ldots \quad (2)$$

Equations (1) and (2) contain three unknown quantities, $\int P dt$, v_1, and ω_1. Another equation can be found by considering the motion of the sphere. Here

$$-P = m_2 \frac{dv}{dt},$$

or
$$-\int P dt = m_2 v \Big]_{u_2}^{V} = m_2 (V - u_2). \quad \ldots \quad (3)$$

This equation introduces still another unknown quantity, V. The last equation necessary for solution is the kinematic equation

$$V = v_1 + \omega_1 h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

EXERCISE 339. Solve equations (1) to (4) for v_1, ω_1, V, and $\int P dt$.

EXERCISE 340. What would be the velocities after the second period of impact in the above discussion if the bodies were (*a*) perfectly inelastic, (*b*) perfectly elastic?

EXERCISE 341. Find the kinetic energy of the above bar after the first period of impact.

EXERCISE 342. If the bar and sphere are perfectly inelastic, calculate the kinetic energy lost by the system after impact.

EXERCISE 343. A uniform slender rod falls through a height h, retaining a horizontal position until one end strikes a fixed obstacle. Assuming the bodies as perfectly inelastic, find the angular velocity of the bar and the linear velocity of the center immediately after impact in terms of h and l, the length of the rod.

EXERCISE 344. If in the preceding exercise the impact takes place at $\frac{1}{4}$ of the length of the rod from one end, find the velocities of the ends of the bar immediately after impact.

Example.—A homogeneous prismatic bar in a horizontal position and constrained to revolve about a vertical fixed axis receives a blow from a sphere whose momentum is $m_2 v_2$. Find the angular velocity of the bar in terms of h, the distance between the point of impact and the mass-center; m_1, the mass of the bar; $\bar{k}$, its principal radius of gyration; and l the distance from the

axis to the point of impact, the bodies to be perfectly inelastic.

Solution.—Fig. 89 illustrates the problem; here A is the fixed axis and C the mass-center of the bar, and I the point of impact.

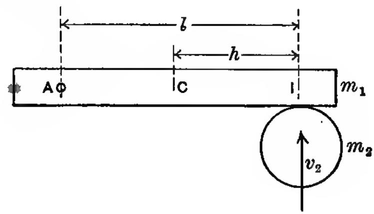


FIG. 89

Let v_1 and ω_1 be the velocities of the bar after impact, then from Fig. 90, which shows the bar as a free body, we have

$$P-Q=m_1\frac{dv}{dt}$$

and
$$Ph+Q(l-h)=m_1\bar{k}^2\frac{d\omega}{dt}.$$

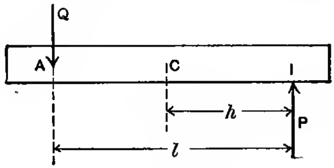


FIG. 90

From the sphere we have

$$-P=m_2\frac{dv}{dt}.$$

Therefore $$\int (P-Q)dt = m_1 v \Big]_0^{v_1} = m_1 v_1, \quad . \quad . \quad . \quad (1)$$

$$\int Phdt + \int Q(l-h)dt = m_1 \bar{k}^2 \omega_1, \quad . \quad . \quad . \quad (2)$$

$$-\int Pdt = m_2 v \Big]_{v_2}^{v_1+\omega_1 h} = m_2(v_1+\omega_1 h - v_2), \quad . \quad (3)$$

and as A remains at rest,

$$v_1 = \omega_1(l-h). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting $\int Qdt = \int Pdt - m_1 v_1$ obtained from (1) into (2) and solving for $l\int Pdt$, we have

$$l\int Pdt = m_1\bar{k}^2\omega_1 + (l-h)m_1 v_1 = m_1\bar{k}^2\omega_1 + (l-h)^2 m_1\omega_1$$

by (4).

But $\int Pdt = m_2(v_2 - v_1 - \omega_1 h)$ from (3), and as

$$v_1 = \omega_1(l-h),$$

we have

$$\omega_1 = \frac{lm_2v_2}{m_1\bar{k}^2 + (l-h)^2 m_1 + m_2 l^2}.$$

Center of Percussion

Assume now that the bar, Fig. 89, were free to translate and that it is required to find where the blow must be struck so that a certain point of the bar (say A) will be the instantaneous axis of rotation. This, of course, precludes any pressure on an axis (imaginary or real) at A.

In Fig. 91 the point of impact so situated that another point A is momentarily at rest is called the *center of percussion*, corresponding to the *spontaneous center of rotation A*.

To find the center of percussion, consider the body as free under the force of impact, P, and remember that A

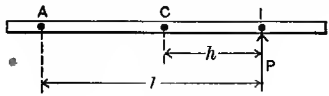


FIG. 91

is to be at rest. Let C (Fig. 91) be the mass-center, m the mass, $\bar{k}$ the principal radius of gyration of the body, and assuming the dimensions shown, we have

$$P = m\frac{dv}{dt}$$

and $$Ph = m\bar{k}^2\frac{d\omega}{dt}.$$

So that $$\int P dt = mv$$

and $$h\int P dt = m\bar{k}^2\omega,$$

where v and ω are the velocities of the bar after impact.

Now $$\int P dt = mv = \frac{m\bar{k}^2}{h}\omega,$$

and $v = \omega(l-h)$, as A is at rest.

$$\therefore\ m\omega(l-h) = \frac{m\bar{k}^2}{h}\omega,$$

or $$(l-h)h = \bar{k}^2.$$

Thus, if $(l-h)$, the distance from the mass-center to the spontaneous center of rotation, and $\bar{k}$, the principal radius of gyration of the body, are known, the distance from the mass-center to the center of percussion, h, can be found.

EXERCISE 345. Show that the center of percussion and its spontaneous center of rotation are convertible and reciprocal.

EXERCISE 346. Given a thin uniform rod 4 feet long, weighing 2 pounds per foot, find the point of impact of a blow struck with the rod so that no jar may be felt at the hand holding the rod (*a*) $\frac{1}{2}$ foot from one end, (*b*) one foot from the end, (*c*) at the end.

EXERCISE 347. If the rod (Ex. 346) is held in the hand and the end strikes a wall, where should the hand be placed so as not to receive a jar?

EXERCISE 348. A pendulum is constructed of a sphere (mass, M; radius, r) attached to the end of a thin rod (mass, m; length, b). Where should it be struck at each oscillation so that there will be no forces due to impact at the point of support?

EXERCISE 349. Find the center of percussion of a sphere which rotates about an axis tangent to its surface.

Ballistic Pendulum

The ballistic pendulum was invented by Robbins, about 1742, for measuring the velocities of bullets. Its use is now entirely superseded by electrical methods. One method of its use was as follows: A rifle is attached in a horizontal position in front of a heavy pendulum which can turn freely about a horizontal axis. The bullet strikes the pendulum and penetrates into it. From

the angle of recoil of the pendulum the velocity of the bullet can be calculated.

Exercise 350. Assume that the time of penetration (impact) is so short that the penetration ceased before the ballistic pendulum moves. Let M be the mass of the pendulum and bullet, m that of the bullet, v the velocity of the bullet just before impact. Also let h be the distance of the center of gravity of the pendulum and bullet, l the distance of the point of impact, and k the radius of gyration of the pendulum and bullet, all measured from the axis of rotation; ω the angular velocity due to the impact and ϕ the maximum angle of recoil. Find v.

PROBLEMS FOR REVIEW

351. A ball thrown up is caught by the thrower 7 seconds afterward. How high did it go, and with what velocity was it thrown? How far below its highest point was it 4 seconds after its start?

352. A ship is sailing north at the rate of 8 miles an hour. In what direction and how fast would a man have to walk on her level deck in order to move at the rate of 7 ft. per sec. towards the east?

353. A fly-wheel moves with an angular retardation of 2 rad.-per-sec. per sec. Its initial velocity was 300 revolutions per minute. How many revolutions will it make before coming to rest?

354. If the radius of the fly-wheel in Exercise 353 is 10 feet, find the acceleration and the velocity of any point in its rim one second after it starts.

355. A particle moves so that

$$x = r\cos t + b,$$
$$y = r\sin t + c.$$

Find (*a*) the rectangular equation of its path,
(*b*) its velocity,
(*c*) its acceleration.

356. Check by the theory of dimensions the equations

$$\omega^2 r = \frac{v^2}{r}$$

and $$a = \omega^2 r \sin t,$$

where the letters have their usual significance.

State clearly which, if any, is incorrect and why.

357. Starting with a differential expression for acceleration deduce an expression for the space between the position of a particle when its velocity is 5 ft. per sec. and its position t seconds later, if the particle moves in a straight line with an acceleration of 2 ft.-per-sec. per sec.

358. A particle moves so that $x = c(t - \sin t)$ and $y = c(1 - \cos t)$.

Find (*a*) the rectangular equation of its path,

(*b*) its velocity,

(*c*) its acceleration.

359. A fly-wheel makes 100 revolutions before coming to rest. If its initial velocity was 120 revolutions per minute, what is its angular acceleration in radians-per-sec. per sec.?

360. (*a*) Deduce from the equations of motion the equation of the trajectory of a particle whose initial velocity and its inclination to the horizontal are 100 ft. per sec. and 60° respectively.

(*b*) From the result of (*a*) find the range on horizontal ground.

361. It is affirmed that

(*a*) Force is the rate of change of energy with respect to time.

(*b*) Force is the rate of change of energy with respect to space.

Test these statements by the theory of dimensions.

362. Starting with the equations of motion of a particle

projected with an initial velocity of 100 ft. per sec. at an angle whose tangent is $\frac{3}{4}$, find

(*a*) the velocity of the particle 1.5 seconds after starting,

(*b*) the range on horizontal ground.

363. Show from the equations of motion of a particle sliding down a curve lying in a vertical plane that the velocity attained is independent of the nature of the curve.

364. Find the centripetal acceleration of a particle moving in a circle of radius r feet when

(*a*) its velocity is v ft. per sec.,

(*b*) it revolves n times per minute,

(*c*) its period is T seconds.

365. A fly-wheel starts with an angular velocity of ω_0 at $t=2$ and has a constant angular acceleration α. Starting with a differential equation, find the angle through which it will turn between $t=2$ and $t=t$ seconds.

366. A particle moves with a constant acceleration a. If its velocity at the time $t=2$ was v_2, find its velocity at any time t and the distance between its position at $t=2$ and its position at any subsequent time t. The results must be derived from the differential equations of motion.

367. A mass of 20 pounds rests upon a horizontal plank, the coefficient of friction between the two being 0.3. What horizontal force is necessary to make the body traverse 8 feet in 2 seconds? What velocity will the mass possess $\frac{1}{5}$ of a second after the start?

368. A weight of 64 pounds rests upon a horizontal board in a railway car and is attached to a spring balance by a horizontal cord. If the coefficient of friction is 0.1 and the car moves with an acceleration of 5 ft. per sec., what will be the reading of the spring balance? After a uniform velocity has been attained, what friction force will act on the weight?

369. A six-inch rapid-fire gun discharges 5 projectiles per minute, each weighing 100 pounds, with a velocity of 2800

feet per second. What kinetic energy do the projectiles possess on leaving the gun? What is the horse-power expended?

370. A spring is stretched 3 feet by a weight of 96 pounds. If this weight be pulled down 2 additional feet and let go, find the time of one complete vibration.

371. A train of 60 tons weight is rounding a curve of radius one mile with a velocity of 20 miles per hour. What is the horizontal pressure on the rails?

372. The weight of a fly-wheel is 8000 pounds and its radius of gyration is 20 feet; the diameter of the axle is 14 inches. The friction force at the axle is 1600 pounds. If the wheel is disconnected from the engine when making 27 revolutions per minute, find how many revolutions it will make before it stops.

373. (*a*) Define: Kinematics, Kinetics, Acceleration, Force, and Momentum.

(*b*) Deduce the dimensions of Acceleration, Force, Momentum, Angular Acceleration, and Normal Acceleration.

374. An elevator, starting from rest, has a downward acceleration of $\frac{1}{2}g$ for 1 second, then moves with constant velocity for three seconds, and then has an upward acceleration of $\frac{1}{3}g$ until it comes to rest.

(*a*) How far does it descend?

(*b*) How much would a 10-pound mass appear to weigh on a spring balance during the descent?

375. (*a*) A weight of 40 pounds is projected along a rough horizontal plane with a velocity of 150 ft. per sec. The coefficient of friction is $\frac{1}{8}$. How far will the weight move in the first five seconds?

(*b*) If the plane in (*a*) be inclined so that its slope is $\frac{3}{4}$, how far would the weight move before coming to rest?

376. (*a*) Define: Centripetal Acceleration.

(*b*) A particle whose weight is 100 pounds revolves about a

vertical axis in a circle whose radius is 4 feet. What force must act upon the particle if the action of gravity is neglected and the particle makes 10 revolutions per second?

(*c*) Describe carefully the direction in which the force computed in (*b*) acts.

377. A fly-wheel whose weight is 2000 pounds and whose radius of gyration is 5 feet revolves at the rate of 100 revolutions per minute. How long will it take a force of 100 pounds, applied tangentially to an axle 2 feet in diameter, to bring the fly-wheel to rest?

378. Find the M. of I. of a right circular cylinder about a diameter of its base as axis. Let the altitude of the cylinder be h and the radius of its base be r.

379. If the M. of I. of a triangular plate, whose base is b and altitude is h, about its base is $\frac{t\delta h^3 b}{12}$, where t is the thickness and δ the density of the plate, find the M. of I. of the plate about an axis through its vertex and parallel to its base.

380. What force must be applied to a body whose weight is 100 pounds to cause it to fall with an acceleration of

(*a*) 40 ft.-per-sec. per sec.?

(*b*) 10 ft.-per-sec. per sec.?

381. (*a*) What is the kinetic energy of a car, weighing 2.5 tons, moving 6 miles per hour and loaded with 36 passengers each of an average weight of 154 pounds?

(*b*) What force would stop this car in the time it would take the car to move 100 feet?

382. Find the horse-power of an engine which is drawing 120 tons up an incline of 1 in 300 at 30 miles per hour against wind and frictional resistances of 20 pounds per ton.

383. A fly-wheel has an angular acceleration of 3 rad.-per-sec. per sec. How many revolutions will it make before its velocity increases from 10 revolutions per minute to 10 revolutions per second?

384. A mass of 200 pounds is acted on by a force of 10 pounds pulling towards the north and a force of 10 pounds pulling towards the east. In what direction will the mass move and how far will it go in one hour if it starts from rest?

385. A body weighing 200 pounds is moved from a state of rest, and is found subsequently to be moving at the rate of 12 ft. per sec. after having traversed 30 feet. If the friction amounted to 30 pounds, what force was pulling the body?

386. A blacksmith's helper using a 16-pound sledge strikes 20 times a minute with a velocity of 100 feet per min. At what rate is he working?

387. From the second law of motion deduce the formula $F=ma$. How is the unit of mass determined? If a force of 10 pounds acts on a mass of 100 pounds, what space will the mass traverse in 5 minutes?

388. State clearly the difference between Work and Power Deduce the dimensions of each. A mass of 10 pounds moves at the rate of 1250 ft. per sec., find the distance through which it would overcome a resistance of 1,000,000 pounds.

389. An inclined plane has a base 120 feet long and is 50 feet high; the coefficient of friction between it and a body weighing 50 pounds placed on it is 0.5. How many units of work are required to draw the body up the plane and how many to draw it down the plane?

390. What must be the effective horse-power of a locomotive which moves at a constant speed of 40 mi. per hr. on level rails, the resistance being 15 pounds per ton, and the weight of engine and train being 100 tons?

If the rails were laid on a gradient 1 to 100, what additional horse-power would be required?

391. Show how the kinetic energy of a system of particles, $m_1, m_2, m_3, \ldots$ all revolving about an axis at radii of r_1, r_2, $r_3 \ldots$ respectively, with an angular velocity ω, may be computed.

392. Compute the radius of gyration of a right circular cone of altitude, a, radius of base, b, about a diameter of its base as axis.

393. A cylinder (mass $= m$, radius $= r$, principal radius of gyration $= k$) rolling upon a horizontal plane has attached to its axis, by means of a horizontal string passing over a pulley, a mass m_1 hanging freely. Find the acceleration of m_1 and the distance it would descend in t seconds starting from rest.

394. The average pressure on the piston of a steam-engine is 60 lbs. per sq. in., the area of the piston is 1 sq. ft., and the length of stroke 18 inches. How many strokes does the engine make per minute if it registers 8 horse-power?

395. Find the dimensions of the constant c in the formula $F = cma$.

396. A fly-wheel weighs 10,000 pounds, and is of such size and shape that the matter composing it may be treated as if concentrated on the circumference of a circle 12 feet in radius. What is its kinetic energy when making fifteen revolutions per minute? How many turns would it make before coming to rest if the steam were cut off and it moved against a friction of 400 pounds exerted on the circumference of an axle 1 foot in diameter? $\left(\pi = \frac{22}{7},\ g = 32.\right)$

397. Define Center of Percussion and Spontaneous Axis of Rotation. Find at what point of a thin uniform rod 4 feet long, weighing 16 pounds per foot, a blow may be struck without jarring the hand holding the rod $\frac{1}{2}$ foot from one end. (Moment of Inertia of rod about axis through center of gravity perpendicular to rod is $m\frac{l^2}{12}$.)

398. A uniform slender bar $4l$ feet long possesses a linear velocity v perpendicular to its length and at the same time an angular velocity $\omega = \frac{v}{l}$ about its center of gravity. Find

the resultant instantaneous velocities of the ends and middle point of the bar in terms of v and l. Which point of the bar is instantaneously at rest?

399. If a conical pendulum be 10 feet long, the half angle of the cone 30°, and the weight of the bob 12 pounds, find the tension of the thread and the time of one revolution.

400. A fly-wheel, external diameter 10 feet and thickness of rim one foot, makes 80 revolutions per minute. Find (a) the angular velocity, (b) the tangential velocity, (c) the normal acceleration of a point on the external and of a point on the internal circumference of the rim, and also of the center of the fly-wheel. Draw diagram showing the directions of these various velocities and accelerations.

401. Two weights of 120 and 100 pounds are suspended by a fine thread passing over a fixed pulley without friction. What will be the velocity of the weights 5 seconds from rest, and what will be the tension of the thread?

402. Find the velocity v of a particle at the distance x from the center of an attractive force, which varies inversely as the square of the distance, if the mass of the particle is m and it starts from rest at a distance a from the center of the force.

403. Define Coefficient of Restitution. Show how we may apply this definition to finding the height to which a body dropped from a height of 64 feet on to a horizontal floor will rise after three rebounds, the Coefficient of Restitution being $\frac{1}{3}$. (Calculation should be carried to two significant figures.)

404. Find the moment of inertia of a rectangular parallelopiped, whose dimensions are a, b, and c, about an axis through its mass-center parallel to the edges whose lengths are a.

405. Find without integration the moment of inertia of the parallelopiped of Exercise 404 about one of the edges, whose length is a, as axis.

406. Find the common velocity after impact of two bodies weighing 2 pounds and 4 pounds and moving in the same direction with velocities of 6 and 9 ft. per sec. respectively.

407. Calculate the kinetic energy lost by impact in Exercise 406.

408. Through what chord of a circle must a body fall to acquire one-half the velocity gained by falling through the diameter?

409. A stone let fall into a well is heard to strike the water in t seconds. Find the depth of the well if at the time of the experiment the velocity of sound was w ft. per sec.

410. A sphere, radius R, oscillates about an axis (a) tangent to the sphere, (b) at a distance of 10 R from the center of the sphere, (c) at a distance of $\frac{R}{2}$ from the center of the sphere. Find the length of the equivalent simple pendulum for each case.

411. Find the principal axes of inertia passing through the mass-center of a rectangular lamina $h \times b$.

412. Find the principal axes of inertia passing through the vertex of the right angle of a lamina in the shape of a right triangle whose legs are a and b.

413. In the motion of a particle down a cycloid prove that the vertical velocity is greatest when it has completed half its vertical descent.

414. A weight of P pounds is drawn up a smooth plane, inclined at 30° to the horizon, by a weight of Q pounds which falls vertically; the weights are connected by a string passing over a pulley at the top of the plane. Find the ratio of P to Q if the acceleration of the weights is $\frac{1}{4}$ the acceleration of gravity.

415. Find the line of quickest descent from the focus to a parabola whose axis is vertical and whose vertex is upwards,

and show that its length is equal to the latus rectum of the curve.

416. Two weights of 5 and 4 pounds together pull one of 7 pounds by means of a string passing over a smooth fixed pulley. After descending through a given space the 4-pound weight is detached without interrupting the motion. Through what fraction of the "given space" will the 5 pounds then descend?

417. An engine and train together weigh 300 tons. Their speed is reduced by pressing brake blocks upon the wheel tires. The maximum pressure upon the blocks is 600 pounds per ton and the coefficient of friction between block and tire is 0.2. How far will the train go while the brakes reduce the speed from (*a*) 50 to 40 miles per hour, (*b*) 40 to 30 miles per hour, (*c*) 30 miles per hour to rest?

418. A 3-ton cage descending a shaft with a velocity of 27 ft. per sec. is brought to rest by a constant force in a space of 18 feet. What is the tension in the supporting cable while stopping the cage?

419. A belt is required to transmit 4 horse-power from a shaft running at 120 revolutions to one at 160 revolutions per minute. Find the stresses in the belt, the small pulley being 2 feet in diameter, and the ratio of the tensions on the belt being as 7 is to 4. Find also the width of belt that would be required in the above case if the stress is taken at 100 pounds per inch of width.

420. Examine the following statement: "Every engineer knows that a thing so balanced as to stand in any position is not necessarily balanced for running; that a 4-pound weight at 3 inches from the axis of rotation though balanced statically by a 1-pound weight at 12 inches from the axis is not balanced by it dynamically. On the contrary, a 4-pound weight at 5 inches is balanced by a 1-pound weight at 10 inches from the axis."

ANSWERS

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1. .5 ft. per sec.; 1969 ft. **2.** Second one.

3. 96800 ft. per sec.

4. 25 ft. per sec.; 17.05 miles per hour.

5. $(32.2)t$, 32.2 ft. per sec.; 193.2 ft. per sec.

6. $3t^2+4t+3$; 23 miles per hour.

7. -32 ft. per sec.

8. 2 ft. per sec.; 332 ft. per sec. **9.** $2bt-at^{-2}$.

10. -1 ft. per sec.; $-\frac{1}{4}$ ft. per sec.

11. ae^t-be^{-t}. **12.** $-\frac{\pi}{2}\sqrt{a^2-s^2}$.

13. .5 ft.-per-sec. per sec.; 108.5 ft. per sec.; 118.5 ft. per sec.

14. 32 ft.-per-sec. per sec.; 224 ft. per sec.

15. 5 sec. after starting; -160 ft. per sec.

16. .733 ft.-per-sec. per sec.

17. 32.2 ft.-per-sec. per sec.; $(6t+4)$ miles-per-hr. per hr.; -32 ft.-per-sec. per sec.; $(6+24t)$ ft.-per-sec. per sec.; $\frac{2a}{t^3}+2b$; $\frac{5}{(4+t)^2}$; ae^t+be^{-t}; $-\frac{a\pi^2}{4}\cos\frac{\pi}{2}t$.

19. $\begin{cases} (a)\ v=t^3-12t^2+32t; \\ \quad\ \ a=3t^2-24t+32. \\ (b)\ 0,\ 4,\ 8. \end{cases}$

20. $5(\log_e^2 b)b^t$.

21. $s=\frac{k}{2}t^2$.

22. 36000 ft.

23. 275 ft.

24. $-20e^{-15}$.

25. $\begin{cases} v=\frac{1}{2}kt^2. \\ s=\frac{1}{6}kt^3. \end{cases}$

35. $\begin{cases} v=10;\ 5;\ 0;\ -5;\ -15;\ -20;\ -30. \\ s=30;\ 37.5;\ 40;\ 37.5;\ 17.5;\ 0;\ -50. \end{cases}$

36. $\begin{cases} v=0;\ 5;\ 10;\ 15;\ 25;\ 30;\ 40. \\ s=-10;\ -7.5;\ 0;\ 12.5;\ 52.5;\ 80;\ 150. \end{cases}$

37. $\begin{cases} v=-20;\ -25;\ -30;\ -35;\ -45;\ -50;\ -60. \\ s=-30;\ -52.5;\ -80;\ -112.5;\ -192.5;\ -240;\ -350. \end{cases}$

38. 21.65 ft. per sec.; 2.86 sec.

39. 95.9 ft. per sec.; 495 ft.; 10 sec.

40. 150 ft.; 200 ft.

41. 5 sec.

42. $\begin{cases} (a)\ v=-gt; & (b)\ v=-gt-V; & (c)\ v=-gt+V; \\ s=-\frac{gt^2}{2}; & s=-\frac{gt^2}{2}-Vt; & s=-\frac{gt^2}{2}+Vt; \\ v^2=-2gs. & v^2=-2gs+V^2. & v^2=-2gs+V^2. \end{cases}$

43. $\frac{V}{g};\ \frac{2V}{g};\ \frac{V^2}{2g};\ -V.$

44. 15600 ft.; 62.5 sec.

45. 114 ft.; 144 ft.

46. 224 ft.

47. 36 ft.; 1 or 2 sec.

48. 8.58 sec.

49. 36 ft.

50. $-\frac{\pi}{18};\ 0;\ \frac{7\pi}{18};\ \pi;\ \frac{29\pi}{18};\ -1.74;\ 0;\ 9.4;\ 0;\ -9.4.$

51. $\frac{1}{18}$.

52. $v=8\cos 4t$; $a=-32\sin 4t$; 0 ft. per sec.; -32 ft.-per-sec. per sec.; 0 ft. per sec. $+32$ ft-per-sec. per sec.

53. 0 ft. per sec.; 0 ft.-per-sec. per sec.

54. Mid-point; at the ends.

56. 5.78 radians; -5 ft.

57. 4.55 ft. per sec.; 1.372 ft.-per-sec. per sec.; 3.14 ft. per sec.; -2.2 ft.-per-sec per sec.

58. 11.67 ft. per sec; 16 ft. per sec.; 4 ft. per sec.

59. 12 ft. per sec.

60. 60°.

61. 2 units in the direction of the velocity of 5 units.

62. .4 miles per hr. $=35.2$ ft. per min.

63. $\begin{cases} \text{15 miles per hr.; } N\tan^{-1}\frac{4}{3}\text{E.} \\ \text{15 miles per hr.; } S\tan^{-1}\frac{4}{3}\text{W.} \end{cases}$

64. 13 miles per hr.

65. 21.9 miles per hr.

66. -2 miles per hr.; 8 miles per hr.; 5.83 miles per hr.

67. 5 miles per hr.

68. 8.48 miles per hr.; N.W. wind.

69. 37.5 miles per hr. **70.** $\begin{cases} v=3 \text{ ft.-per-sec.; } 180°; \\ a=6 \text{ ft.-per-sec. per sec.; } 270°. \end{cases}$

71. $\begin{cases} x=a\cos^{-1}\dfrac{a-y}{a}-\sqrt{2ay-y^2}. \\ y=x\tan\theta-\dfrac{gx^2}{2v^2\cos^2\theta}. \end{cases}$

72. (*b*) 88.5 ft. per sec.; 11° 45′; +32 ft.-per-sec. per sec.; downward.

73. $v=r$; $a=r$.

79. $\begin{cases} x=6t^2;\ y=\frac{4}{3}t^3-9t;\ y=(\frac{2}{3}x-9)\sqrt{\dfrac{x}{6}}; \\ v=4t^2+9;\ v_n=0;\ v_t=4t^2+9; \\ a_x=12;\ a_y=8t;\ a=\sqrt{144+64t^2}; \\ a_n=12;\ a_t=8t; \\ \rho=\dfrac{(4t^2+9)^2}{12};\ s=\frac{4}{3}t^3+9t. \end{cases}$

80. 201.2 ft. per sec.; −30°; 10106.8 ft.-per-sec. per sec.; 60°.

81. $\begin{cases} (a)\ x=r\sin\omega t; \\ \quad\ y=r\sin\omega t. \\ (b)\ x=r\sin\omega t \\ \quad\ y=r\sin\left(\omega t-\dfrac{\pi}{2}\right). \\ (c)\ x=r\sin\omega t; \\ \quad\ y=r\sin(\omega t-\pi). \end{cases}$

82. $\begin{cases} x=r\sin\omega t; \\ y=r\sin\left(\omega t+\dfrac{\pi}{6}\right). \end{cases}$

83. $\begin{cases} x=r\sin\omega t \\ y=2r\sin\left(\omega t+\dfrac{\pi}{6}\right). \end{cases}$

84. $\begin{cases} x=r\sin\omega t; \\ y=\dfrac{r}{2}\sin(\frac{3}{2}\omega t). \end{cases}$

91. .4795; 1.558; .1411; .2837; −.587.

92. 52.2 radians per sec. **94.** 28.6 rev. per min.

93. 141.4 radians per sec. **95.** (20*t*) radians per sec.; 0; 200.

96. −2 radians-per-sec. per sec.

97. $\omega=\pi\cos\pi t$; $\alpha=-\pi^2\sin\pi t$; $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; $-\pi^2$; $+\pi^2$; $-\pi^2$.

100. ($2\pi n$) rad. per sec.

101. 300 radians per sec.; 14320 rev. **102.** 250 rev.

103. 12.56 radians-per-sec. per sec. **104.** πdn ft. per min.

105. 44 rad. per sec.; 66 ft. per sec.; 2900 ft.-per-sec. per sec.

106. $\frac{v}{r}$; $2v$; 0.

109. 27.1 ft. per sec.; 50.8 ft.-per-sec. per sec.; 20.4 ft. per sec.; 34.5 ft.-per-sec. per sec.

110. -10 ft. per sec.; 30 ft. per sec.; 10 ft. per sec.; 1 ft. from center.

111. (*b*) $2r\omega$ ft. per sec.

112. $30\sqrt{2}$ ft. per sec.; 453.0 2 ft.-per-sec. per sec.; 60 ft. per sec.; 450.04 ft.-per-sec per sec.

114. $x^2+y^2=c^2$.

116. 2 ft.

118. 10.25 ft. per sec.

119. 3.11.

120. 1 unit; 32 lbs.

121. 12 lbs.

122. 37 ft. per sec.; 420 ft.

123. 3 sec.

124. 6.71 sec.

125. 6.56 lbs.

126. 6 lbs. [MLT^{-2}.

128. LT^{-2}; MLT^{-1}; MLT^{-1};

129. $M^{\circ}L^{\circ}T^{\circ}$.

130. T^{-1}; T^{-2}; LT^{-2}.

136. The dyne $=\frac{15.432}{7000\times 30.48}$ poundals.

139. 1440 lbs.; 1350 lbs.; 1440 lbs.

140. 2940 lbs.; 2756 lbs.; 2940 lbs.

141. 2.14 ft.-per-sec. per sec.; 13.7 sec.

142. 153.6 ft. per sec.; 4608 ft.

143. 6875 lbs.; 1980 ft. **144.** 197 lbs.; 300 lbs.

145. 103 lbs.; 0 lbs.; -37.5 lbs.

146. 150 lbs.

147. 20520 ft.

148. 25 ft. per sec.

150. 57.8 ft. per sec.

151. 11250 lbs.

153. $y=x\tan\theta-\frac{gx^2}{2v_0^2\cos^2\theta}$.

155. 45°.

159. 3.13 ft.; 12.5 ft.

160. 2.16 sec.; 95.3 ft.

161. 18.6 ft.

162. 13.9 ft.

163. 15°; 75°.

164. $y=\frac{v_0^2}{2g}-\frac{gx^2}{2v_0^2}$.

165. $\dfrac{2u^2 \cos^2 \alpha \tan \beta}{g}$.

166. 10° 18′; 490 ft. per sec.

167. 33° 41′; 77° 54′.

168. $v = -a\sqrt{\dfrac{c}{m}} \sin\left(\sqrt{\dfrac{c}{m}}t\right)$.

170. $\begin{cases} a_y = -64y. \\ v_y = 2\sqrt{1-16y^2}. \\ y = \frac{1}{4}\sin\left(8t + \dfrac{3\pi}{2}\right). \end{cases}$

171. $\dfrac{\pi}{4}$ sec.

172. $\begin{cases} a_y = -\dfrac{gy}{e}; \\ v_y = \sqrt{\dfrac{g}{e}}\sqrt{a^2-y^2}; \\ y = a\sin\left\{\sqrt{\dfrac{g}{e}}t + \dfrac{\pi}{2}\right\}. \end{cases}$

173. $t = \sqrt{\dfrac{m}{c}}\left\{\log\left(\dfrac{s+\sqrt{s^2-a^2}}{a}\right)\right\}$.

174. 7 miles per sec.

175. $t = \dfrac{s_0}{R}\sqrt{\dfrac{s_0}{2g}}\left\{\sin^{-1}\sqrt{\dfrac{s}{s_0}} + \dfrac{1}{s_0}\sqrt{s(s_0-s)}\right\}$.

176. $\pi\sqrt{\dfrac{m}{c}}\left(\dfrac{b}{2}\right)^{\frac{3}{2}}$.

178. $4\frac{4}{7}$ ft.-per-sec. per sec.; $109\frac{5}{7}$ lbs.; $73\frac{1}{7}$ lbs.

179. 2 lbs.

180. 32 ft. per sec.

181. $\frac{32}{3}$ ft.-per-sec. per sec.; 133 lbs.; 100 and 200.

182. 1 sec.

183. 60 ft. per sec.; 13.8 lbs.

184. It will.

185. 25 m.

186. $m(25+g)$.

187. 100 lbs.; 324 lbs.; 990 lbs.; 14.3 lbs.

188. 24800 lbs.

189. .157 sec.

190. 3490000 lbs.

191. 95000−64; 95000+64 lbs.

192. $\dfrac{4w\pi^2n^2l}{g}$ lbs.

193. 89° 57′; 10 ft.; $4000\pi^2m$.

194. 5.9 lbs.

195. $\tan^{-1}\dfrac{v^2}{gr}$.

196. 5.6″.

197. 5° 7′.

198. (.111m) lbs.; 32.201 ft.-per-sec. per sec.

199. $\dfrac{4\pi^2 R \cos^2\theta}{T^2}m.$

203. $\sqrt{2gk+v_0{}^2}$; $\sqrt{2g(k-2r)+v_0{}^2}$; $k+\dfrac{v_0{}^2}{2g}$; $\sqrt{2g(2r-y)}$.

205. $5W$; $6W$.

206. $v^2=2g(k-y)$; $\rho=\{8r(2r-y)\}^{\frac{1}{2}}$; $R=mg\dfrac{2r+k-2y}{\sqrt{2r(2r-y)}}$.

207. $2mg$.

208. $t=\sqrt{\dfrac{2}{g}}\left[\text{vers}^{-1}\dfrac{2y}{k}\right]_0^k=\pi\sqrt{\dfrac{r}{g}}$.

211. 1.108; 1.108; 1.113.

212. 1.1077 sec.

213. 32.13 ft.-per-sec. per sec.

214. 108.

215. 32.08.

218. .016 ft. per sec.; 115.2 ft.

219. 28.7.

220. ML^2T^{-2}; ML^2; *No.*

222. 1.5 radians-per-sec. per sec.; 6 radians per sec.; 1720 rev.

223. 86 rev. per min.

224. 12.9 rev. per min.

225. 1.11 radians-per-sec. per sec.; 10 radians-per-sec. per sec.; .4 radians-per-sec. per sec.

226. $\dfrac{\pi n I}{30 W\mu r}$.

227. $\dfrac{\pi n^2 I}{3600 W\mu r}$.

228. $\dfrac{\delta a l^3}{3}$.

229. $\dfrac{\delta b l}{3}(3a^2+l^2+3lc+3c^2)$.

232. $\dfrac{h^3bt\delta}{12}$; $\dfrac{hb^3t\delta}{12}$.

233. $\dfrac{b^3ht\delta}{3}$; $\dfrac{h^3bt\delta}{3}$.

234. $\dfrac{t\delta h^3 b}{12}$.

235. $\dfrac{\delta ch^3}{12}(b_2+3b_1)$; $\dfrac{\delta c}{3}\{h^3b-(h-t)^3(b-t)\}$; $\dfrac{\delta c}{288}\{h^4-t^4+48th^3+4t^3(6b-11t--h)\}$.

236. $\dfrac{al^3\delta}{12}$.

237. $\dfrac{t\delta h^3 b}{36}$.

238. $t\delta hb\left(\dfrac{h^2}{12}+a^2\right)$.

241. $\dfrac{t\delta\pi r^4}{2}$.

243. $\dfrac{t\delta hb}{12}(h^2+b^2)$.

244. $\dfrac{l}{\sqrt{3}}$; $\dfrac{1}{\sqrt{3}}\sqrt{3a^2+l^2+3lc+3c^2}$.

245. $\frac{b^2}{12}$; $\frac{h^2}{12}$; $\frac{h^2}{6}$. **247.** $\frac{\delta t h^3 b}{12 r^2}$. **248.** π; 4π; $\frac{\pi}{25}$.

249. 3600 lbs.; 900 lbs.; 9 lbs.; 28.1.

255. $\alpha = \frac{gr}{k^2+r^2}$; $T = \frac{k^2 gm}{k^2+r^2}$. **256.** $\frac{grt^2}{2(k^2+r^2)}$.

257. .01866 radians-per-sec. per sec.; 1.119 radians per sec.

258. .00466 radians-per-sec. per sec.; .28 radians per sec.

259. .0186 radians-per-sec. per sec.; 1.116 radians per sec.

260. 99.7 lbs. **261.** $\frac{5mg}{(2M+5m)r}$.

262. $\alpha = \frac{g(rm_2 - Rm_1)}{mk^2 + m_1 R^2 + m_2 r^2}$.

263. $\sqrt{\frac{Wr\sqrt{2}}{I+\frac{W}{g}r^2}}$; $\sqrt{\frac{2Wr}{I+\frac{W}{g}r^2}}$; 0; $\sqrt{-\frac{Wr\sqrt{2}}{I+\frac{Wr^2}{g}}}$.

264. $\alpha = \frac{gr(1+2\cos\theta)}{k^2+r^2(1+\cos\theta)^2}$. **276.** $\sqrt{\frac{98}{101}}$.

267. 8.2 ft.; 1.59 sec. **277.** 0; 500 ft.-lbs.

270. 1.64 ft. **278.** 10440 ft.-lbs.

271. 1.72 sec. **279.** ML^2T^{-2}.

272. 1.54 sec. **280.** $W(b\mu - h)$.

273. $2\sqrt{\frac{lg\sin\beta}{3}}$; $\frac{2}{r}\sqrt{\frac{lg\sin\beta}{3}}$. **281.** 88000 ft.-lbs.

283. 32400 ft.-lbs.

274. $\frac{2}{3}$.

275. $\tan^{-1} 3\mu$. **284.** $\frac{Wl}{2}$ ft.-lbs.

285. ML^2T^{-2}; ML^2T^{-2}; ML^2T^{-3}.

286. 63.1 H.P.: **287.** 69300 gal. **288.** 31.2 H.P.

290. 90 ft.-lbs.; 2.5 ft.-lbs. **292.** 9 ft.-lbs.; 25 ft.-lbs.

293. 400; 256; 144; 64; 16; 0. 0; 144; 256; 336; 384; 400.

294. 7750 ft.-lbs; 450 ft.-lbs.; 300 ft.-lbs.

295. 1:3.

296. $\frac{W\pi^2n^2d^2}{2g}$.

298. 30250 ft.-lbs.

299. $\mu = .163$.

302. If the radii are equal $\frac{\text{hoop}}{\text{disk}} = \frac{4}{3}$.

303. 244 lbs. **304.** 1920000 lbs. **305.** 2.24 ft.; 60.35 ft.-lbs.

306. 10 tons; 10.4 tons.

307. 10083 ft.

308. 84.5.

309. $\frac{W(r+\mu r')}{r-\mu r'}$.

310. $\frac{2R}{r_1-r_2}$.

311. $1875\left(\frac{r_1-r_2}{R}\right)$.

314. 1.62 lbs.

318. 67.6 lbs. per sq. in.

319. 38.2.

320. 42.5 H.P.

321. .76 H.P.

322. $\frac{2\pi nWl}{33000}$.

323. 656.

327. 53° 8′; 45°.

328. 24 ft. per sec.; 9 ft.

329. 30°.

330. $\tan^{-1}\sqrt{\frac{e^3}{e^2+e+1}}$.

333. -1.1 ft. per sec.; 1.9 ft. per sec.

334. $m_1 = \frac{48e+37}{11}m_2$.

335. $v_2 = -\frac{u_2}{4}$; $v_1 = +\frac{u_2}{4}$.

336. $\frac{3}{5}$.

337. 123 ft.

339.
$$\begin{cases} \int P\,dt = \dfrac{m_1 m_2 \bar{k}^2 u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m^2}; \\[2ex] v_1 = \dfrac{m_2 \bar{k}^2 u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2}; \\[2ex] \omega_1 = \dfrac{m_2 h u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2}; \\[2ex] V = \dfrac{m_2 (\bar{k}^2 + h^2) u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2}. \end{cases}$$

340. $$\begin{cases} u = \dfrac{m_2(\bar{k}^2 + h^2) - e m_1 \bar{k}^2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2} u_2; \\[2ex] v_2 = \dfrac{(1+e) m_2 \bar{k}^2 u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2}; \\[2ex] \omega_2 = \dfrac{(1+e) m_2 h u_2}{\bar{k}^2 m_1 + (\bar{k}^2 + h^2) m_2}. \end{cases}$$

341. $\dfrac{m_1 v_1{}^2}{2} + \dfrac{m_1 \bar{k}^2 \omega_1{}^2}{2}$.

343. $\omega = \dfrac{3\sqrt{2gh}}{2l}$; $v = \frac{3}{4}\sqrt{2gh}$.

344. Up, $\frac{3}{7}\sqrt{2gh}$; downward, $\frac{9}{7}\sqrt{2gh}$.

346. $\frac{26}{9}$; $\frac{10}{3}$; $\frac{8}{3}$ ft. from end.

347. $\frac{4}{3}$ ft. from other end.

348. $\dfrac{M\{\frac{2}{5}r^2 + (b+r)^2\} + \frac{1}{3}mb^2}{M(b+r) + \frac{1}{2}mb}$ from the point of support.

349. $\frac{7}{5}R$.

350. $v = \dfrac{(M+m)h}{m}\omega = \dfrac{(M+m)h}{m}\sqrt{\dfrac{2gh}{k^2}(1 - \cos\phi)}$.